IMBEDDING OF HIGHER TYPE THEORIES.

by

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1. INTRODUCTION

In his address to the Nice Congress in 1970 and later in his lecture at the Oslo Symposium in June 1972 G. Sacks defined an abstract 1-section to be a countable admissible set M satisfying local countability and Δ_0-dependent choice, and he proved that a set M is an abstract 1-section if and only if for some normal type-2-functional F, $1-sc(F) = M \cap \mathcal{P}(\omega)$. His proof is found in Sacks [18].

I later gave another proof for this theorem, see Normann [17]. The starting point for the theory described in this paper was an attempt to imitate the proof of [17] in a setting of higher types. I found it convenient to work within the general context of computation theories. The notion of a computation theory was introduced by Moschovakis [15] and was further developed by Fenstad [4]. A good reference for our purpose is the recent survey Fenstad [5].

I was successful in carrying through the first step viz to imbed a higher type computation theory in a suitable set theoretic structure (see section 1.1), but I did not succeed in lifting the forcing argument. However, there are other applications of the imbedding theory which will be described in sections 1.3 and 1.4.

I would like at this point to thank J. Moldestad for his everlasting willingness to explain the "hard" facts of higher type

recursion theory to me. I will also express my thanks to J.E. Fenstad for many helpful discussions and suggestions concerning this paper.

1.1. The general imbedding

Companion theory for theories over ω is well developed. Let Θ be a theory in which ω is finite, and let Θ be p-normal. (We define finiteness and p-normality in 1.5. See Fenstad [5] for details.) Then $1\text{-en}(\Theta)$ forms a Spector-class.

Let $R_\Theta = \{\langle a,\sigma\rangle;\ a$ is a Θ-computation and $|a|_\Theta = \sigma\}$. Let $\alpha = \sup\{|a|_\Theta\ ;\ a$ is a Θ-computation$\}$. Then $\langle L_\alpha^{R_\Theta}, R_\Theta\rangle$ is called the companion of Θ. We then have the following results:

<u>Theorem A</u> Let M be the companion of Θ.

<u>a</u> $1\text{-sc}(\Theta)$ consists exactly of those subsets of ω that are Δ_1-definable over M, i.e.

$$1\text{-sc}(\Theta) = M \cap {}^{\omega}2$$

<u>b</u> $1\text{-en}(\Theta)$ consists exactly of those subsets of ω that are $\Sigma_1(R_\Theta)$-definable over M.

<u>c</u> $\langle M,R_\Theta\rangle$ is an admissible structure, Σ_1-projectible to ω

<u>d</u> $x \in M$ if and only if there is a subset $A \subseteq \omega \times \omega$, $A \in M$, such that A is isomorphic to $\langle TC(\{x\}),\in\rangle$

<u>e</u> All elements of M are countable inside M, and M satisfies Δ_0-dependent choices. □

<u>Remark</u> The basic source for this can be found in the construction of meta recursion theory by Kreisel and Sacks [13], and the construction of the 'next admissible' by Barwise, Gandy and Moschovakis [2]. For further information and technical development,

see Sacks [18] and Moschovakis [16]. A general imbedding theorem for Specter-theories is also stated in Fenstad [4].

Now, let $I = \bigcup_{i \leq k} tp(i)$ for some $k \in \omega$.

Let Θ be a computation theory on I satisfying:

i ω is the domain of codes and the values of computations.

ii I is weakly Θ-finite.

iii Θ is p-normal.

iv c_i (= the characteristic function for type (i)) is Θ-computable for all $i \leq k$
$ev^{(i)}$ (= the point-evaluation on type i, $ev^{(i)}(\alpha^i, \alpha^{i-1}) = \alpha^i(\alpha^{i-1})$) is Θ-computable.

We have a Θ-computable pairing $\langle\, , \rangle$, with Θ-computable inverses, and a function $(\ )_i$ such that $\forall \{x_i\}_{i\in\omega} \exists y \in I\, (y)_i = x_i$, being Θ-computable. These functions are called primitives.

v There is a Θ-index e such that

$$\{e\}(x,e') = x(\lambda b\{e'\}(b,x))$$

whenever this makes sense.

Then we call Θ a type-k-theory. For further information, see the forthcoming paper by Fenstad [5].

We are going to imbed the full section of Θ in a family of structures $\langle M_a\rangle_{a\in I}$ in V_I in the following way (V_I = the universe of sets with I as urelements. See Barwise [1]):
We say that $A \subseteq I \times I$ is a code for a set $x \in V_I$ if

i $A/\simeq$ is isomorphic to $TC(\{x\})$, where $a \simeq b$ $\Leftrightarrow \langle a,b\rangle \in A \;\&\; \langle b,a\rangle \in A$.

ii Pairs $\langle 0,a\rangle$ in the domain of A will always code the ur-element $a \in TC(\{x\})$, and $a \in TC(\{x\})$ will always and only be coded by $\langle 0,a\rangle$.

$M_a[\Theta]$ is defined to be those elements in V_I having a code in $k+1-sc\,\Theta[a]$, where

$$\langle e,\sigma,n\rangle \in \Theta[a] \iff \langle e,a,\sigma,n\rangle \in \Theta .$$

We call $\langle\langle M_a[\Theta]\rangle_{a\in I}, R_\Theta\rangle$ the spectrum of Θ, and denote it $\mathrm{Spec}(\Theta)$.

Theorem B

a Each M_a is countable (lemma 3.6).

b Each M_a is rudimentary closed in L_Θ and our primitives. (lemma 3.7).

c Each M_a satisfies $\Delta_0^{R_\Theta}$-separation (lemma 3.8).

d Each M_a satisfies $\Delta_0^{R_\Theta}$ - DC (lemma 3.13).

e If $x \in M_a$, then $x \subseteq M_a$ if and only if x is countable in M_a. (lemma 3.14).

$\square$

Let $M = \bigcup_{a\in I} M_a$. Let $P \subseteq M$.

We say that P is Σ^*-definable if there is a Δ_0-formula φ without parameters such that for all a and all $x \in M_a$,

$$x \in P \iff \exists y\,\varphi(x,y) \iff (\exists y \in M_a)\,\varphi(x,y)$$

If P and $M \setminus P$ both are Σ^*, then P is called Δ^*-definable. This notion may be relativized to R_Θ and to an $a \in I$.

Theorem C

a (Σ^*-collection. For proof see lemma 3.12).

Let $a \in I$. Assume φ is $\Delta_0^{R_\Theta a}$ and

$\forall b \in I \; \exists x \in M_{\langle a,b\rangle} \varphi(x,b)$.

Then there is a set $u \in M_a$ such that

$\forall b \in I \; \exists x \in u \cap M_{\langle a,b\rangle} \varphi(x,b)$

b Each M_a is uniformly Σ_1-projectible to ω. $\square$

Theorem D (Theorem 3.17)

Let Θ be a theory on I, $A \subseteq I$.

Then $A \in k+1-\mathrm{en}(\Theta) \Longleftrightarrow A$ is Σ^*-definable over $\mathrm{Spec}(\Theta)$. $\square$

Remark In Harrington [7], Harrington proves a theorem which is essentially the same as theorem D.

Corollary (Corollary 3.18)

Let Θ_1 and Θ_2 be two theories, $\mathrm{Spec}(\Theta_1) = \langle M_a\rangle_{a\in I}$, $\mathrm{Spec}(\Theta_2) = \langle N_a\rangle_{a\in I}$. Then the following are equivalent

i Θ_1 and Θ_2 are equivalent theories.

ii $(\forall a \in I)\,(M_a = N_a)$ and R_{Θ_1} and R_{Θ_2} are Δ^* in each other.

$\square$

1.2 Abstract Approach

Let $\langle N_a\rangle_{a\in I}$ be a set of structures in V_I. $\langle N_a\rangle_{a\in I}$ is called a family if

i $I \in N_a$ for all $a \in I$.

ii $(\forall a,b \in I)\,(a \in N_b \Longleftrightarrow N_a \subseteq N_b)$.

By an abstract spectrum we mean a family $\langle N_a\rangle_{a\in I}$ and a relation $R \subseteq N = \bigcup_{a\in I} N_a$ such that i For all $a \in I$ and all $x \in V_I$, $x \in N_a$ if and only if x has a code in N_a.

ii Each N_a is rudimentary closed.

iii $\langle N_a \rangle_{a \in I}$ satisfies $\Sigma^*(R)$-collection.

iv $N_a = \Delta^*_a(R)$, where we use no parameters but natural numbers or a.

v $N_a \subseteq L^R_{\mathrm{rank}(M_a)}[I]$.

Theorem E

a (Theorem 4.7).

$\langle \langle N_a \rangle_{a \in I}, R \rangle$ is an abstract spectrum if and only if there is a theory Θ such that $\mathrm{Spec}(\Theta) = \langle \langle N_a \rangle_{a \in I}, R_\Theta \rangle$ and R and R_Θ are Δ^* in each other over $\langle N_a \rangle_{a \in I}$.

b (Theorem 4.6).

A family is called nice if it satisfies i-iii above. If $\langle N_a \rangle_{a \in I}$ is a nice family relative to F, where F is a functional of type $k+2$, then $(\forall a \in I)\ (k+1\text{-}\mathrm{sc}(F,a) \subseteq N_a)$.

$\square$

D.B. McQueen proved in [14] a special case of b in the theorem above.

Theorem F (Theorem 4.4)

Let $\langle M_a \rangle_{a \in I}$ be a nice family. Let f be a function defined on $M = \bigcup_{a \in I} M_a$ such that for all $a \in I$, when $x \in M_a$, then $f(x) \in M_a$ (i.e. f is closed in $\langle M_a \rangle_{a \in I}$). Assume that (the graph of) f is Δ^*. Define the function Γ inductively by

$$\Gamma(\gamma) = f(\langle \Gamma(\beta) \rangle_{\beta < \gamma})$$

Then Γ is Δ^* definable, and if $\gamma \in M_a$, then $\Gamma(\gamma) \in M_a$ and $\langle \Gamma(\beta) \rangle_{\beta<\gamma} \in M_a$.

Corollary (corollary 4.5)

The function $\Gamma(p,\gamma) = L^p_\gamma[I]$ is Δ^*-definable, and if $\gamma \in M_a$, $p \in M_b$ then $L^p_\gamma[I] \in M_{\langle a,b \rangle}$.

$\square$

1.3 Recursion in a normal type k+2 object.

Let Θ, $\langle M_a \rangle_{a \in I}$ and R_Θ be given. We say that $\langle M_a \rangle_{a \in I}$ is R_Θ-impenetrable if for all functions g such that $\forall a \forall x \in M_a g(x) \in M_a$ and such that the graph of g is Δ^*-definable, there is a nice family relative to $R_\Theta, \langle N_a \rangle_{a \in I}$ such that $\forall a (N_a \subseteq M_a)$, and for some a, $N_a \neq M_a$, and g is closed in $\langle N_a \rangle_{a \in I}$.

This definition generalizes the following way of characterizing recursively Mahlo:

M is Mahlo if for all Δ_1-functions g there is an admissible $N \subsetneq M$ such that $g''N \subseteq N$.

Theorem G (Theorem 6.3)

The following statements are equivalent:

i There is a normal type k+2-functional F such that $k+1-\mathrm{en}(\Theta) = k+1-\mathrm{en}(F)$.

ii $\mathrm{Spec}(\Theta)$ is not R -impenetrable.

$\square$

Remark Over ω this result was proved by Simpson [10], and independently by Kechris and Harrington [9]. In the setting of higher types a result essentially equivalent to theorem G was independently proved by Kechris [11].

The following theorem which recently has been proved by L. Harrington and J. Moldestad complements the characterization result of the previous theorem.

Theorem H.

Let Θ be a type-k theory and assume that tp(k-1) is strongly Θ-finite (i.e. the semirecursive relations are closed under $\exists\alpha^{k-1}$).

Then there is a normal type-k+2 functional F such that

i $k+1-sc(F,a) = k+1-sec(\Theta[a])$, for all $a \in I$.

ii $k-en(F) = k-en(\Theta)$. $\square$

Remark. Sacks [19] proved i in the case where Θ is the type-k-theory associated with a normal functional G of type $\geq k+3$. Harrington [7] proved ii under the same assumptions. Theorem H gives a characterization since tp(k-1) is strongly finite in the recursion theory derived from a normal type-k+2 functional (see e.g. MacQueen [14]).

1.4 Recursion in the Super-jump.

The Superjump was introduced by R. Gandy [6]. L. Harrington defined in [7] a notion of strong recursion in S^{k+3}, F^{k+2}. We call the resulting computation theory the Harrington-theory for S^{k+3}, F. Harrington proved that this will be a type-k-theory in our sense.

Our next result was inspired by the main results of Harrington [8].

a Let ρ^F be the ordinal for recursion in S^3, F i.e. $\rho^F = \omega_1^{S^3,F}$. Then ρ^F is the 1^{st} recursively Mahlo in F ordinal.

b $L^F_{\rho^F} \cap \mathcal{P}(\omega) = 1 - sc(S^3.F)$.

Now, let Θ, $\langle M_a \rangle_{a \in I}$ and R_Θ be given. We say that $\langle M_a \rangle_{a \in I}$ is strongly R_Θ-impenetrable if for all $a \in I$, all Δ^*_a-functions being closed in the family $\langle M_{\langle a,b \rangle} \rangle_{b \in I}$, there is a nice family relative to R_Θ, $\langle N_b \rangle_{b \in I}$ such that for all b .

$$N_b \subsetneq M_{\langle a,b \rangle} .$$

We say that $\langle M_a \rangle_{a \in I}$ is hyper impenetrable in R_Θ if $\langle M_a \rangle_{a \in I}$ is strongly impenetrable in R_Θ, and for all Δ^*-functions g being closed in $\langle M_a \rangle_{a \in I}$, g is closed in a proper, strongly impenetrable in R_Θ, subfamily $\langle N_a \rangle_{a \in I}$ of $\langle M_a \rangle_{a \in I}$.
Over ω, this gives us hyper-Mahlo.

Theorem I (theorem 7.7)

Let Θ be a type-k-theory. The following two statements are equivalent:

i There is a type k+2 functional F such that $k+1-en(\Theta)$ = the envelope of the Harrington theory of S^{k+3},F .

ii Spec(Θ) is strongly R_Θ-impenetrable, but not hyper impenetrable in R_Θ .

□

Over ω we then have:

Let α be the ordinal of Θ . Θ is the Harrington-theory of S^3 and some type-2 functional F if and only if α is R_Θ Mahlo but not R_Θ-hyper Mahlo.

1.5 Notations

The set of finite-type functionals over ω is defined by $tp(0) = \omega$ = the set of natural numbers.

$tp(k+1) = {}^{\omega}tp(k)$ = all functions defined on $tp(k)$ with values in ω.

Kleene [12] defined a hierarchy for recursion in higher types. We will use an equivalent hierarchy presented in Fenstad [5].

Let $I = tp(0) \cup \ldots \cup tp(k)$ for a given $k \in \omega$.

A computation theory Θ on I is called finite if there is a Θ-code e such that

$$\{e\}(e',a) = \begin{cases} 0 & \text{if } \forall b \{e'\}(a,b) \simeq 0 \\ 1 & \text{if } \{e'\}(a,b) \text{ converges for all } b \text{ , and } \\ & \quad \exists b \{e'\}(a,b) \simeq 1 \text{ .} \end{cases}$$

To all Θ-computations a, there is an ordinal $\|a\|_{\Theta}$, indicating the length of the computation.

Θ is a p-normal theory if there is a Θ-computable function $\{e\}$ such that

If a or b are Θ-computations, then $\{e\}(a,b)$ converges, i.e. has a value.
In that case

$$\{e\}(a,b) = \begin{cases} 1 & \text{if } \|a\|_{\Theta} \leq \|b\|_{\Theta} \\ 0 & \text{if } \|a\|_{\Theta} > \|b\|_{\Theta} \end{cases}$$

By convention, $\|b\|_{\Theta} = \infty$ when b is not a computation.

We will use some standard notation, among which we mention

$\{e\}(a)\downarrow$ = the computation has a value.

$\{e\}(a)\uparrow$ = the computation never stops.

$\{e_1\}(a) \simeq \{e_2\}(b)$. The values are both defined and equal.

V_I The universe of sets having I as urelements, i.e. the result of iterated power-set operation on I .

TC transitive closure operator.

Aknowledgements

Much of this theory is independantly developed by Y.N. Moschovakis, A.S. Kechris and L. Harrington. Analogues to the theorems B - E and theorem G are also proved by them.

2. A METHOD OF CODING

In this section we will construct a way of coding sets in V_I as subsets of I and study some of the properties of this coding. Most of the proofs are rather technical and involve no new methods. They are therefore given in detail in an Appendix, since we believe that they do not contribute essentially to the understanding of the main body of the text.

On I we have a recursive pairing function, $\langle\, ,\, \rangle$, and recursive partial inverses $(\ )_0$ and $(\ )_1$. For $i \leq k$, let c_i be the characteristic function of $tp(i)$, i.e.

$$c_i(x) = \begin{cases} 0 & \text{if } x \in tp(i) \\ 1 & \text{otherwise} \end{cases}$$

Let $ev^{(i)}$ be the evaluation function defined on $tp(i) \times tp(i-1)$ as follows

$$ev^{(i)}(a,x) = a(x)$$

Given an ω-sequence $\langle a_i \rangle_{i\in\omega}$ from I, we may code it as one element

$$a = \langle a_i \rangle_{i\in\omega}$$

such that the inverse function

$$f(i,a) = (a)_i$$

is recursive. Note that the definition may be made so that $(a)_0$ and $(a)_1$ are the same, irrespective of whether a is looked upon as a pair or as a sequence.

In this part we will regard I as a set of urelements, and the functions described above as primitives.

In the ω-situation, recall part d of theorem A of the introduction. There we had a set A , recursive in the theory Θ , and isomorphic to $\langle TC\{X\},\in\rangle$. We might call A a code for X .

It is this way of coding sets we want to generalize. The coded sets are to be taken from the universe V_I .

There are two natural problems:

a If we claim isomorphism between a code B and the structure $\langle TC(\{X\}),\in\rangle$, we would have no control over which sets we might code from the $k+1$ - section of a given recursion theory. If $V = L$, we have nice wellorderings, while under other set-theoretic assumptions, there are no computable well-ordering of length $\aleph_1$.

Also, uniqueness would create great technical problems.

b We may have $\langle TC(\{X\}),\in\rangle$ isomorphic to $\langle TC(\{Y\}),\in\rangle$, without having $X = Y$, e.g. if $X = a$ and $Y = b$, where a and b are different elements of I .

Concerning b , it is not hard to see that if $\langle TC(\{X\}),\in\rangle$ is isomorphic to $\langle TC(\{Y\}),\in\rangle$ with an isomorphism being the identity when restricted to the urelements, then $X = Y$.

We solve problem b by always letting an urelement a be coded by $\langle 0,a\rangle$, and always letting $\langle 0,a\rangle$ code a .

We solve problem a by identifying an element $Y \in TC(\{X\})$ not only by a single urelement, but by a block of urelements. This means for instance that an ordinal α will be coded by prewell-orderings of length $\alpha+1$, not only wellorderings.

Our formal definition will be

<u>Definition 2.1</u>

A partial function

$$\rho : \mathcal{P}(I) \to V_I$$

is defined as follows:

$\rho(A)$ is defined if

<u>i</u> A is a set of pairs such that no pairs of the form $\langle a, \langle 0, b\rangle\rangle$ is in A, where $a \neq \langle 0, b\rangle$.

<u>ii</u> There is a set $X \in V_I$ and a function

$$f : \text{dom } A \xrightarrow{\text{onto}} TC(\{X\})$$

such that

(*) $(\forall a, b \in \text{dom } A)(f(a) \in f(b) \iff \langle a, b\rangle \in A \ \& \ \neg(\langle b, a\rangle \in A))$
$\& \ (f(a) = f(b) \iff \langle a, b\rangle \in A \ \& \ \langle b, a\rangle \in A))$

(**) If f is restricted to the pairs $\langle 0, a\rangle \in \text{dom } A$, f becomes a 1-1-map, and this restriction is a function onto the urelements of $TC(\{X\})$ such that

$$(\forall \langle 0, a\rangle \in \text{dom } A)(f\langle 0, a\rangle) = a)$$

By dom A we mean $\{(a)_0 , a \in A\} \cup \{(a)_1 ; a \in A\}$.

If such X and f exist, they will, by trivial verifications, be unique, and we let

$$\rho(A) = X . \qquad \square$$

If A is a code, and $a \in \text{dom } A$, we say that a codes $f(a) \in TC(\{X\})$.

Note that a code for an ordinal α is a prewellordering of length $\alpha+1$ in $I \setminus (\{0\} \times I)$.

Remark Through section 2 we call a set 1.order definable if it is either a subset of I , or a subset of $\mathcal{P}(I)$, and if it is defined by a formula arithmetic over I , in which we may use our primitives. Thus all quantifiers shall be taken over I .

Lemma 2.2

The set of codes is first order definable over I and in our primitives.

A proof is found in the Appendix.

□

In the next lemmas, we will see that definable manipulations with sets often may be 'translated' to first order manipulations with codes.

Lemma 2.3

a Let A be a code for a set x . Then we may, by a first order formula over I and our primitives, define a code B for the function

$$f: A \to TC(\{x\})$$

described in definition 2.1.

b Let A_1 and A_2 be codes for ordinals α_1 and α_2 . Then we may, by a first order formula, define a code for the ordinal $\alpha_1 + \alpha_2$.

The proof is found in the Appendix.

□

A natural closure property on a structure is to require closure under rudimentary functions.

For the notion of rudimentary functions, see for instance K. Devlin [3]

In our definition, we add the primitives to the standard definition. Then we obtain:

Definition 2.4

a A function $f: V_I^n \to V_I$ is called rudimentary if it is generated by the following schemas:

i. $f(x_1 \ldots, x_n) = x_i \qquad 1 \leq i \leq n$

ii. $f(x_1, \ldots, x_n) = x_i \setminus x_j \quad 1 \leq i,\ j \leq n$

iii. $f(x_1, \ldots, x_n) = \{x_i, x_j\} \quad 1 \leq i,\ j \leq n$

iv. $f(x_1, \ldots, x_n) = \bigcup_{y \in x_1} h(y, x_2, \ldots, x_n)$

where h is rudimentary.

v. $f(x_1, \ldots, x_n) = h(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n))$

where $h, g_1, \ldots, g_m$ are rudimentary.

vi. The functions $(\ )_i$, $\langle , \rangle$, c_i , and $ev^{(i)}$ are rudimentary.

b $M \subseteq V_I$ is rudimentary closed if

$(\forall \vec{x}, f)(\vec{x} \in M^n$ & f is rudimentary $\Rightarrow f(\vec{x}) \in M)$

where $\vec{x} = \langle x_1, \ldots, x_n \rangle$ denotes a sequence.

c Let $M \subseteq V_I$

Rud Cl(M) = The unique rudimentary closure of M

= The least X such that $M \subseteq X$ and X is rudimentary closed.

d Let $F \subseteq V_I$. If we add schema

vii. $f(x) = x \cap F$.

We obtain rud_F , rudimentary relative to F . □

From now on in part 2, F will always be a functional of type $k+2$. Then the graph of F will be a subset of V_I.

A formula φ is called Δ_0 if it is a formula in the usual language for V_I, with $\in$ and the primitives as the only relation symbols, and in which all quantifiers are bounded. See Barwise [1].

<u>Lemma 2.5</u>

<u>a</u> Let φ be a Δ_0-formula relative to our primitives on I. Then there is a rudimentary function f_φ such that

$$f_\varphi(\vec{x}) = \begin{cases} \langle\vec{x}\rangle & \text{if } \varphi(\vec{x}) \\ \emptyset & \text{otherwise} \end{cases} .$$

If φ is Δ_0^F, f_φ is rud_F.

<u>b</u> Let $M \in V_I$. If A is first order definable over $\langle M, \in, (\ )_i, \langle\ ,\ \rangle, ev^{(i)}, c_i\rangle$ then there is a rudimentary function f_A, depending only on the definition of A, such that $f_A(M) = A$.

Moreover, if A is first order in F, f_A will be rud_F.

We prove that <u>b</u> follows from <u>a</u>, and leave the proof of <u>a</u> for the Appendix.

Let A be defined by the formula ψ.

Let $\varphi(M,x) = \psi_M(x)$; i.e. ψ with all quantifiers restricted to M. φ is Δ_0. Let f_φ be the function from part <u>a</u>. Define

$$f(M) = \bigcup_{x\in M}\{(f_\varphi(M,x))_1\}$$

$$= \{x;\ f_\varphi(M,x) = \langle M,x\rangle\} .$$

(Here $\langle\ ,\ \rangle$ and $(\ )_1$ have their usual set-theoretic meaning).

For all elements in M , except $\emptyset$, it is clear that

$$\Psi_M(x) \iff x \in f(M) .$$

It is trivial to modify the definition such that

$$\Psi_M(\emptyset) \iff \emptyset \in f(M) \quad \text{if necessary.}$$

□

Our next definition will enable us to handle codes more freely. The intention is that when we have a sequence of codes, we want to treat the members in a uniform way.

If A_1 and A_2 are two different codes, $a_1 \in \operatorname{dom} A_1$ and $a_2 \in \operatorname{dom} A_2$, we cannot by a first order analysis normally check if a_1 and a_2 codes the same or different sets.

Thus we will sometimes restrict ourselves to a situation where the same elements of I codes the same elements in V_I , independent of which code in the sequence we choose.

We have also seen, in some of the proofs, that we often need an extra supply of elements to be able to create new codes.

Both these needs are taken care of in the next definition:

<u>Definition 2.6</u>

Let $A_1,\ldots,A_n$ be a sequence of codes, let $f_1,\ldots,f_n$ be the associated functions from definition 2.1.

We say that $A_1,\ldots,A_n$ is a <u>compatible</u> sequence of codes if

<u>i</u>. $(\forall i,j \leq n)(\forall a \in I)(a \in \operatorname{dom} A_i \,\&\, a \in \operatorname{dom} A_j \Rightarrow f_i(a) = f_j(a))$

<u>ii</u>. $(\forall y)(\forall i,j \leq n)(\forall a \in I)(a \in \operatorname{dom} A_i \,\&\, f_i(a) = y \,\&\, y$ is coded in A_j by some element $\Rightarrow a \in \operatorname{dom} A_j \,\&\, f_j(a) = y$.

<u>iii</u>. $(\forall i \leq n)(\exists n_i < \omega)(\forall a \in \operatorname{dom} A_i)(a$ is a pair $\langle k,b\rangle$ where $k \leq n_i \,\&\, b \in I)$.

□

A single code A is called compatible if it satisfies <u>iii</u>.

<u>Lemma 2.7</u>

Compatibility is first order definable.

The proof is in the Appendix.

$\square$

<u>Lemma 2.8</u>

There is a function f' such that whenever A is a compatible code for a set x, $a \in \text{dom } A$ and a codes $y \in TC(\{x\})$, then $f'(A,a)$ is a code for y, compatible with A. By that we mean that $\{A, f'(A,a)\}$ is a compatible sequence.

Moreover, $f'(A,a) = \varphi(I,\omega,A,a)$ for some rudimentary function φ.

We find it convenient to give the proof here as it is quite simple.

A code b in A is a code for a set $z \in TC(\{y\})$ if and only if there is a finite sequence $a_1,\dots,a_n$ such that $a_1 = b$, $a_n = a$, and for all $i \leq n$, $\langle a_i, a_{i_1}\rangle \in A$.

The formal definition will be

$$\langle b,c\rangle \in f'(A,a) \iff (\exists d \in I)(\exists n \in \omega)(\forall i \leq n)(\langle (d)_i,(d)_{i+1}\rangle \in A \;\&\; (d)_o = b \;\&\; (d)_1 = c \;\&\; (d)_n = a) .$$

By lemma 2.5.b this is rudimentary.

$\square$

In the next lemma we prove that rudimentary functions on sets may be imitated as rudimentary functions on compatible sequences of codes.

<u>Lemma 2.9</u>

Let f be a rudimentary function, $f\colon V_I^n \to V_I$. Let ρ be the function from definition 2.1.

Then there is a rudimentary function f^* such that whenever $A_1,\ldots,A_n$ is a compatible sequence of codes, then $f^*(A_1,\ldots,A_n)$ is a code compatible to $A_1,\ldots,A_n$, and

$$\rho(f^*(A_1,\ldots,A_n)) = f(\rho(A_1),\ldots,\rho(A_n)) .$$

Moreover, if F is of type $k+2$ and f is rud_F, then f^* will be rud_F.

The proof is by a careful induction on the defining schema for f. This induction is given in the Appendix, and we content ourselves here by calling attention to the following fact:

Remark. Compatibility is not transitive. For instance, given x_1,x_2 and x_3 with codes A_1,A_2 and A_3, we want to find a code for $x_3\setminus(x_1\setminus x_2)$.

Our code B for $x_1\setminus x_2$ is compatible with A_1,A_2 and A_3. Then we forget about A_1 and A_2 and find a code (for $x_3\setminus(x_1\setminus x_2)$) compatible with B and A_3. Note that C need not be compatible with A_1 and A_2. $\square$

In the proof of lemma 2.9 we define the notion of "main member of a code A", being an element of $\mathrm{dom}(A)$ coding $\rho(A)$. We also define the rudimentary function g_3, having the following property:

If A is a code for a set x, then $g_3(A)$ consist of exactly the urelements $b \in \mathrm{dom}\, A$ coding an element of x.

These two concepts will be used later in the text. (See lemmas 3.12 and 3.13). $\square$

Lemma 2.10

Let M be the class of sets coded by elements in $\mathcal{P}(I)$. Let Ψ be a Δ_0-formula without parameters. Then there is a first order definable subset A_Ψ of $\mathcal{P}(I)^n$ such that

$$\vec{A} \in A_\Psi \iff \vec{A} \text{ is a sequence of compatible codes and } M \models \Psi(\rho(\vec{A}))$$

where we by $\rho(\vec{A})$ mean $\langle \rho(A_1),\dots,\rho(A_n)\rangle$

This may be relativized to F .

The proof is by induction on the length of the formula Ψ , and is left for the Appendix. □

Remark. The M in lemma 2.10 will consist of those elements of V_I having hereditary cardinality less than or equal to that of I.

Up to this point, we have worked with codes for sets and manipulations with these.

Now we will study substructures M of the universe V_I in which there is a nice correspondence between the elements of M and the codes in M .

Definition 2.11

Let $\mathcal{Y} \subseteq \mathcal{P}(I)$. By the structure of $\mathcal{Y}$ we mean

$$\{x: (\exists A \in \mathcal{Y})(A \text{ is a code and } \rho(A) = x\}.$$

Thus $\mathrm{Str}(\mathcal{Y}) = \rho''(\mathcal{Y})$.

$M \subseteq V_I$ is called an abstract structure if for all $x \in V_I$ we have

$$x \in M \iff x \text{ has a code in } M .$$ □

Note that $M \subseteq V_I$ is an abstract structure if and only if

$$M = \mathrm{Str}(M \cap \mathcal{P}(I))$$

We see from lemma 2.3 that if M is a rudimentary closed abstract structure, then M will have rather strong closure properties.

For instance, let $\alpha_1, \alpha_2 \in On$. Then

$$\alpha_1 \in M \ \& \ \alpha_2 \in M \Rightarrow \alpha_1 + \alpha_2 \in M .$$

Proof Let $A_1 \in M$ be a code for α_1 , $A_2 \in M$ be a code for α_2 . By lemma 2.3 there is a code A_3 for $\alpha_1 + \alpha_2$, first order definable from A_1 and A_2 . But then there is a rudimentary function f such that $f(A_1, A_2) = A_3$.
Thus $A_3 \in M$, since M is rudimentary closed.

But $A_3 \in M$, and A_3 is a code for $\alpha_3 = \alpha_1 + \alpha_2$.
Then $\alpha_3 \in M$ itself. $\square$

Remark. It seems likely that rudimentary closed abstract structures are closed under primitive recursive ordinal functions.

Note that rudimentary closed abstract structures need not be transitive.

How are we then going to interpret the various kinds of formumas? Our main principle is that bounded quantifiers shall be absolute, while unbounded quantifiers shall be interpreted dependent of the actual structure.

To make this precise, we need a special language.

Definition 2.12

Let $M \subseteq V_I$. The language $\mathcal{L}_M$ will consist of the following symbols:

i Constant symbols $\underline{x}$ for each $x \in M$.

Subconstant symbold $\underline{\underline{x}}$ for each $x \in TC(M)$.

A list of variables.

ii Negation, $\neg$, and disjunction, $\vee$.

iii $\in$, $=$, $\langle , \rangle$, $(\ )_i$, c_i , $ev^{(i)}$.

iv Bounded quantifiers $\exists v \in x$.

v Unbounded quantifiers.

By a sentence we mean a closed formula without subconstant symbols.

We interpret the formulas as follows:

$\underset{\sim}{x}$, $\underset{\sim}{y}$ will denote constant symbols or subconstant symbols.

i $M \models \underset{\sim}{x} \in \underset{\sim}{y}$ if $x \in y$.

$M \models \underset{\sim}{x} = \underset{\sim}{y}$ if $x = y$

Analougs for $\langle , \rangle$, $(\ )_i$, c_i and $ev^{(i)}$.

ii $\neg$ and $\vee$ are interpreted as usual.

iii Unbounded quantifiers $\exists v$

$$M \models \exists v\, \varphi(v) \iff (\exists x \in M)(M \models \varphi(\underline{x}))$$

iv Bounded quantifiers

$$M \models \exists v \in \underset{\sim}{x}\, \varphi(v) \iff (\exists y \in x) M \models \varphi(\underline{\underline{y}}) \ .$$

□

The distinction between constant symbols and subconstant symbols is important when we want a formula to be interpretable in substructures of M .

Note the difference between iii and iv. A bounded quantifier may range over something outside M , while an unbounded quantifier do range over M .

To illustrate, note that when M is not transitive, the two formulas are not always equivalent:

$$(\exists v)(v \in \underset{\sim}{x} \,\&\, \varphi(v)) \quad \text{and} \quad (\exists v \in \underset{\sim}{x})\, \varphi(v) \ .$$

The nontransitivity of M has caused some problems, which excuses the emotional overtones of the following definition.

<u>Definition 2.13</u>

Let $M \subseteq V_I$, and let $x \in M$.
We say that <u>x is clean</u> (M-clean) if $x \subset M$.
We say that <u>x is dirty</u> (M-dirty) if x is not clean.

□

3. THE SECTION AND ENVELOPE OF A THEORY

The coding technique of section 2 will now be used to imbed type-k-theories into suitable set-theoretic structures.

Again we leave some of the proofs for the Appendix, either because they are tedious, but fairly trivial technical proofs, or because we do not need the results for applications in this paper, but for their own sake and, may be, for later use.

Definition 3.1

By a type-k-theory Θ we mean a computation theory over I such that

i ω is the domain of codes and the values of computations.

ii I is weakly Θ-finite, i.e. there is an index e such that whenever e' is an index, then

$$\{e\}(a,e') = \begin{cases} 0 & \text{if } \lambda b\{e'\}(a,b) \text{ is total and} \\ & \text{for some } b \ \ \{e'\}(a,b) = 0 \\ 1 & \text{if } \lambda b\{e'\}(a,b) \text{ is total, but} \\ & \text{for no } b \ \ \{e'\}(a,b) = 0 \\ \text{Undefined otherwise} & \end{cases}$$

iii Θ is p-normal (i.e. there is a Θ-computable functional p such that

$$p(\sigma,\delta)\downarrow \quad \text{if } \sigma \text{ or } \delta \text{ is a } \Theta\text{-computation,}$$

and then

$$p(\sigma,\delta) = 0 \quad \text{if } |\sigma|_\Theta < |\delta|_\Theta$$

$$p(\sigma,\delta) = 1 \quad \text{if } |\sigma|_\Theta \geq |\delta|_\Theta .$$

Here $|\ |_\Theta$ is the length function associated with the theory Θ.

<u>iv</u> $\langle\,,\rangle$, $(\;)_i$, c_i and $ev^{(i)}$ are Θ-computable.

<u>v</u> There is a Θ-index e such that

$$\{e\}(x,e') = x(\lambda b\{e'\}(b,x))$$

whenever this makes sense.

<u>vi</u> All computations are single valued. □

By $\Theta[a]$ we mean the following theory

$$(e,\sigma,x) \in \Theta[a] \iff (e,a,\sigma,x) \in \Theta \;.$$

When Θ is a type-k-theory, $\Theta[a]$ will be a type-k-theory for all $a \in I$.

<u>Remark</u> For further information on computation theories see e.g. Fenstad [5].

The existence of a selection operator for numbers will be basic for our further theory. This was proved in various degrees of generality by Gandy, Moschovakis and Grilliot. For a recent exposition, see McQueen [14].

<u>Lemma 3.2</u>

Θ admits a <u>selection operator for numbers</u>, i.e. if $A \subseteq \omega \times I$ is Θ-semicomputable, then there is a Θ-computable function f such that

$$\forall a(\exists n(\langle n,a\rangle \in A) \Rightarrow f(a)\downarrow \;\&\; \langle f(a),a\rangle \in A)$$ □

As a simple consequence of weak finiteness we obtain that the set of semicomputable relations in I is closed under universal quantification over I .

<u>Lemma 3.3</u>

Recursion in a normal object F of type $\geq k+2$ form a theory over I.

This result is basically due to R. Gandy $\square$

Note that we did not demand that $tp(k-1)$ is strongly Θ-finite, i.e. that the Θ-semicomputable relations are closed under existential quantifiers over $tp(k-1)$.

D.B. McQueen proved this in [14] when Θ is the type-k-theory derived from some normal type $k+2$ functional. It is not known if $tp(k-1)$ is strongly Θ-finite for arbitrary type-k-theories.

Definition 3.4

Let Θ be a type-k-theory. By the full section of Θ , we mean those subsets of I that are in $k+1-sc(\Theta[a])$ for some $a \in I$. $\square$

Recall the definition

$$R_\Theta = \{\langle a,\alpha\rangle \; ; \; a \text{ is a } \Theta\text{-computation and } |a|_\Theta = \alpha\} .$$

Note that the full section of Θ is exactly those subsets of I constructible in R_Θ over I up to $\mathrm{rank}(R_\Theta)$. This was observed by Harrington in [7] .

Now we are going to define the structure in which we will imbed recursion in Θ .

Definition 3.5

Let Θ be a theory.
Let $\mathcal{Y}_a[\Theta]$ be the $k+1$-section of the theory $\Theta[a]$.
Let $M_a[\Theta] = \mathrm{Str}(\mathcal{Y}_a[\Theta])$ (See definition 2.11).

We call $\langle\langle M_a[\Theta]\rangle_{a\in I}, R_\Theta\rangle$ the spectrum of Θ , and denote it by $\mathrm{Spec}(\Theta)$. $\square$

We will omit $[\Theta]$ when it is clear from the context. We will sometimes also denote the family $\langle M_a\rangle_{a\in I}$ by $\mathrm{Spec}(\Theta)$.

Remarks A set A is in the $k+1$ - sc $\Theta[a]$ if and only if there is a Θ-computable function f, not necessarily total, such that $\lambda x f(a,x)$ is the characteristic function of A. A will then have a Θ-index.

Thus a set x will be an element of M_a if and only if there is a Θ-index, e, such that $\lambda b(\{e\}_{\Theta}(a,b))$ is the characteristic function of a code for x.

In theorem A in the introduction we gave several characterizations of the companion of a theory over ω. Definition 3.5 is a generalization of characterization d from this theorem. We will later prove that the original definition of the companion also in a suitable way generalizes to Spec(Θ).

Lemma 3.6

Each M_a is a countable abstract structure (definition 2.11). The proof is trivial and may be found in the Appendix.

A general result of computation theories is the second recursion theorem. It states:

Let F be a Θ-computable functional. Then there is a Θ-computable function f such that for all $a \in I$

$$F(f,a) \simeq f(a)$$

f is called a fix-point for F, and we sometimes refer to this theorem as the fix-point-theorem.

In most of our applications, there will only be one unique fix-point which will be total. Mostly f will be the characteristic function of some code.

Lemma 3.7

Each M_a is rudimentary closed in R_{Θ}.

Proof By lemma 2.9 it suffices to prove

i $x \in M_a \Rightarrow x \cap R_\Theta \in M_a$.

ii If $x_1,\dots,x_n \in M_a$, then we may find a compatible sequence $A_1,\dots,A_n$ of codes for $x_1,\dots,x_n$ in M_a .

We prove i and leave the proof of ii, which uses the fix-point-theorem, for the Appendix.

Let $x \in M_a$, Let $A \in M_a$ be a code for x . A is a well-founded relation with length $|A|$. We claim that there is a computation with argument a of length greater than $|A|$. If not, we would be able to prove that the set of computations in a would be a-computable.

Let α be the length of such a computation. Then

$$R_\Theta \cap x = R_\Theta \restriction \alpha \cap x .$$

But $R_\Theta \restriction \alpha = \{\langle b,\beta\rangle \,;\, \beta < \alpha \,\&\, \langle b,\beta\rangle \in R_\Theta\}$ is an element of M_a . By ii, M_a is rudimentary closed, and i follows. □

Recall definition 2.12 about interpretation of formulas on nontransitive structures. By that definition a Δ_0-formula holds in M_a if and only if it holds in the universe.

Lemma 3.8

M_a satisfies Δ_0-separation , i.e.

If φ is a Δ_0-formula without parameters, and $x \in M_a$, then

$$\{y \in x;\ \varphi(y)\} \in M_a$$

Proof By lemma 2.5 there is a rudimentary function f_φ such that

$$f_\varphi(y) = \begin{cases} y & \text{if } \varphi(y) \\ 0 & \text{otherwise} \end{cases} .$$

Let $g(x) = \bigcup_{y \in x} \{f_\varphi(y)\}$

Since M_a is rudimentary closed, $g(x) \in M_a$.

Now

$$\{y \in x;\ \varphi(x)\} = g(x)$$

or

$$\{y \in x;\ \varphi(x)\} = g(x) \setminus \{\emptyset\}.$$

In both cases, the set will be in M_a. $\square$

The proof is general, so if any structure M is rudimentary closed, it will satisfy Δ_0-separation.

Of course lemma 3.8 may be relativized to R_Θ.

By Δ_0-Collection we mean the following principle:

Let φ be a Δ_0-formula, $u \in M_a$.

Assume

$$M_a \models \forall x \in u \ \exists y\ \varphi(x,y).$$

Then there is a M_a-clean set v such that

$$\forall x \in u \ \exists y \in v\ \varphi(x,y)$$

By **Δ_0-Dependent Choices** (Δ_0-DC) we mean the following principle:

Let φ be a Δ_0-formula.

Assume

$$M_a \models \forall x \ \exists y\ \varphi(x,y).$$

Then there is a sequence $\langle x_i \rangle_{i \in \omega} \in M_a$ such that

$$(\forall i \in \omega) M_a \models \varphi(x_i, x_{i+1})$$

Before we are able to verify these principles, we need some more machinery!

Definition 3.9

We say that $x \in \omega$ is an a-index for a set $C_x \subseteq I$ if

$$\{x\}_{\Theta[a]}(b) = \begin{cases} 0 & \text{if } b \in C_x \\ 1 & \text{if } b \notin C_x \end{cases}$$

Lemma 3.10

Let Ψ be a Δ_0-formula, ρ the function from definition 2.1. Let $a \in I$.

Then there is a subset $B_\Psi[a]$ of ω, uniformly $\Theta[a]$-semi-computable, such that whenever $\vec{x}$ is a sequence of a-indices for sets $\vec{C}_x$, then

$$\langle\vec{x}\rangle \in B_\Psi[a]$$

$\iff$ $\vec{C}_x$ is a compatible sequence of codes and $\Psi(\rho(\vec{C}_x))$.

Proof By lemma 2.10 there is a set A_Ψ in $(\mathcal{P}(I))^n$ that is first order definable over I, such that

$\vec{A} \in A_\Psi$ if and only if $\vec{A}$ is a compatible sequence of codes such that $\Psi(\rho(\vec{A}))$.

A_Ψ is obviously Θ-computable, and there will be a number e such that whenever $\vec{x}$ are a-indices for $\vec{C}_x$ we have

$$\{e\}_\Theta(a,\vec{x}) = \begin{cases} 0 & \text{if } C_{\vec{x}} \in A_\Psi \\ 1 & \text{if } C_{\vec{x}} \notin A_\Psi \end{cases}.$$

Let $B_\Psi[a] = \{\vec{x};\ \{e\}_\Theta(a,\vec{x}) \simeq 0\}$. $\square$

Lemma 3.11

The relation 'x is an a-index for a set' is $\Theta[a]$-semi-computable, uniformly in a.

Proof

x is an a-index for a set

if and only if

$\{x\}_{\Theta[a]}$ is the characteristic function of a set

if and only if

$$(\forall y \in I)(\exists z \in \omega)((z = 0 \vee z = 1) \ \& \ \{x\}_{\Theta[a]}(y) \simeq z) .$$

By definition, almost,

$$\{\langle x,y,z\rangle ; \ \{x\}_{\Theta[a]}(y) \simeq z\}$$

is $\Theta[a]$-semicomputable.

By the selection theorem for numbers, the semicomputable relations are closed under existensial number quantifiers.

Since I is weakly Θ-finite, the semicomputable relations are closed under universal quantification over I . $\square$

Lemma 3.12

$M_a \models \Delta^{R_\Theta}$-Collection.

Proof

Let $M_a \models \forall x \in u \, \exists y \, \varphi(x,y,R_\Theta,\vec{x})$, where $u,\vec{x} \in M_a$ and φ is Δ_0 .

Given $u,y,\vec{x}$, we know that all information we need about R_Θ lies inside $TC\{u,y,\vec{x}\}$. Thus we are only interested in $R_\Theta \upharpoonright \max \operatorname{rank}(u,y,\vec{x})$.

Now, assume that we have given a code A for u and codes $\vec{B}$ for $\vec{x}$, with fixed a-indices.

Given an a-index e for a code for a set y, we may, uniformly in e , by the fix-point-theorem, find an index for a code for $\max \operatorname{rank}(u,y,\vec{x})$.

Again, using the fix-point theorem, we find an a-index for a code for

$$R_{\Theta} \upharpoonright \mathrm{rank}(u,y,\vec{x}) \ .$$

Let g be the function computing this index from e.

Let $b \in \mathrm{dom}\, A$ code an element $x_b \in u$. $b \in g_3(A)$, where g_3 is the rudimentary function from the proof of lemma 2.9, picking out just that kind of b's. Let

$$C = \{\langle b,e\rangle;\ e \text{ is an index for a set } y \text{ , and}$$
$$\varphi(x_b, y, R_{\Theta} \upharpoonright \mathrm{rank}\{u,y,\vec{x}\},\vec{x})\} \ .$$

By lemmas 3.10 and 3.11 we see that C is semicomputable.

By the selection theorem for numbers, lemma 3.2, there is a computable function f, defined on $g_3(A)$ such that

$$(\forall b \in g_3(A))(\langle b,f(b)\rangle \in C) \ .$$

Let $C_{f(b)}$ be the code with index $f(b)$. Using the fix-point-theorem as in lemma 3.7, we may assume that the codes are compatible, and that $\langle 1,0\rangle$ is not in the domain of any of them. Let

$$C_u = \bigcup_{b \in g_3(A)} C_{f(b)} \cup \{\langle d,\langle 1,0\rangle\rangle;\ d \text{ is the main member of some}$$
$$C_{f(b)}\} \cup \{\langle\langle 1,0\rangle, \langle 1,0\rangle\rangle\} \ .$$

Recall from the proof of lemma 2.9 that a main member in a code A for a set x is a $b \in \mathrm{dom}\, A$ coding x. C_u is $\Theta[a]$-computable, and it is a code for exactly the kind of set we wanted to construct. $\square$

By standard methods we may prove that M_a satisfies Δ_1-separation and Σ_1-collection. We use Δ_0-separation and collection.

<u>Lemma 3.13</u>

$M_a \models \Delta_0^{R_\Theta} - DC$.

The proof is of the same kind as that of lemma 3.12, and is left for the Appendix. □

We say that a structure M satisfies <u>clean countability</u> if all M-clean elements of M are countable inside M.

<u>Lemma 3.14</u>

For all $a \in I$, M_a satisfies clean countability.

For proof, see the Appendix. □

Δ_0-DC and local countability played a key rôle in the proof of Sack's abstract 1-section result. The notion of clean countability is intended as an analogue of the notion of local countability and may be useful in the search for an abstract $k+1$- section result.

As an introduction to our next concept, Σ^*-definability, we prove that each M_a is, uniformly in a, $\Sigma_1(R_\Theta)$-projectible to ω.

Define

$$\begin{aligned} p_a(x) = n \iff \exists\alpha(&\{b;\ \langle\langle n,a,b,0\rangle,\alpha\rangle \in R_\Theta\} \text{ is a code for } x \\ &\& \ \forall\beta<\alpha\ \forall m\in\omega\,\{b;\langle\langle m,a,b,0\rangle,\beta\rangle \in R_\Theta\} \text{ is not a code for } x \\ &\& \ \forall m<n\ (\{b;\ \langle\langle m,a,b,0\rangle,\alpha\rangle \in R_\Theta\} \text{ is not a code for } x\ )\ . \end{aligned}$$

The matrix here is trivially seen to be Δ_1 , and since for a given x the n above exists and is unique, we have found a projection.

Up to this point we have only discussed the properties of each M_a separately. But it is the family $\langle M_a\rangle_{a\in I}$ that describes the full section and the envelope of Θ.

Let $M = \bigcup_{a\in I} M_a$.

We say that a subset $P \subseteq M$ is Σ^*_b-<u>definable</u> if there is a Δ_0-formula φ without parameters such that whenever $x \in M_{\langle a,b\rangle}$

(*) $x \in P \iff (\exists y \in M)\varphi(x,y) \iff (\exists y \in M_{\langle a,b\rangle})\varphi(x,y)$.

Moreover, P is Σ^*-definable if P is Σ_0^*-definable; se definition 2.12 for interpretation of the formulas.

In most cases we allow extra relation symbols, $\underline{R}_1,\ldots,\underline{R}_n$, in the Δ_0-formula φ . We then obtain $\Sigma_b^*(R_1,\ldots,R_n)$ and $\Sigma^*(R_1,\ldots,R_n)$, where $R_1,\ldots,R_n$ are the interpretations of $\underline{R}_1,\ldots,\underline{R}_n$.

We use the equivalence between the two formulas in (*) in the proof of lemma 3.16. The essential part of (*) is

$$x \in P \iff (\exists y \in M_{\langle a,b\rangle})\varphi(x,y) .$$

By Σ^*-collection we mean the following principle:

Let $b \in I$ and let $\vec{x}$ be a sequence of sets from M_b . Let φ be a Δ_0-formula. If

$$\forall a \in I \ \exists x \in M_{\langle a,b\rangle}\varphi(a,x,\vec{x})$$

then there is a u in M_b such that

$$(\forall a \in I)(\exists x \in u)\varphi(a,x,\vec{x}) .$$

We prove that $\langle M_a\rangle_{a\in I}$ satisfies $\Sigma^*(R_\Theta)$-collection exactly in the same way that we proved ordinary Δ_0-collection: By the selection theorem for numbers we find a function f , Θ-computable in b such that $f(a)$ is an index for a set in $M_{\langle a,b\rangle}$ satisfying φ . We make the codes compatible, and glue them together.

Instead of finding a set u such that

$$\forall a \in I \ \exists x \in u \ \varphi(a,x,\vec{x})$$

we might find a function $h \in M_b$ such that

$$\forall a \in I \ \varphi(a,h(a),\vec{x}) .$$

The advantage of this is that for all such functions, $h(a) \in M_{\langle a,b\rangle}$.

We could do the same with ordinary collection. Instead of a clean set, we construct a function defined on ω. That a set is a function defined on ω may be expressed by a Δ_0-formula, while cleanness is only Σ_1-definable.

$P \subseteq M$ is Δ^* if both P and $M \setminus P$ are Σ^*. As a corollary of Σ^*-collection, we obtain Δ^*-separation.

The following lemma gives some useful closure properties of the class Δ^*:

<u>Lemma 3.16</u>

<u>a</u> If P is Δ^*-definable, and Q is $\Delta_0(P)$, then Q is Δ^*-definable.

<u>b</u> If P is Δ^*-definable and $b \in I$, then $P \cap TC(M_b)$ is Δ_1-definable over M_b.

A detailed proof is given in the Appendix. □

In the proof of <u>a</u> we use the absoluteness properties of the formulas. These properties do not hold for ordinary Δ_1-definitions over M_b. Nevertheless, we do have $\Delta_0(\Delta_1) \subseteq \Delta_1$ over M_b. The main trick in the proof (which we omit) is to use clean countability.

Note that the Σ^*-sets are not always closed under bounded existensial quantification. Let φ be a Δ_0-formula, and let $A \subseteq I$ be defined by

$$b \in A \iff \exists a \in I \, \exists x \, \varphi(a,b,x)$$

and suppose that φ has the correct absoluteness property, i.e. if we find an x, then we may find it in $M_{\langle a,b\rangle}$.

We may also define A by

$$b \in A \iff \exists x \, \exists a \in I \, \varphi(a,b,x) .$$

We cannot however in this case always find the x in M_b. Hence the absoluteness property is lost.

The Σ^*-formulas are closed under universal quantification. The proof of this is almost identical to the proof of the inclusion $\Delta_0(\Delta^*) \subseteq \Delta^*$.

We see that the Σ^*-subsets of I have important properties in common with the semicomputable relations. This leads up to the following theorem:

Theorem 3.17

Let Θ be a type-k-theory and $A \subseteq I$. Then

$$A \in k+1-\mathrm{en}\ \Theta$$

if and only if

$$A \text{ is } \Sigma^*\text{-definable in } R_\Theta .$$

Proof

Let A be semi-computable;

$$A = \{a;\ \{n\}(a) \simeq 0\} .$$

Define

$$P(x,a) \Longleftrightarrow x \in R_\Theta \ \&\ (x)_0 \text{ is a triple}$$
$$\&\ ((x)_0)_0 = n \ \&\ ((x)_0)_1 = a \ \&\ ((x)_0)_2 = 0$$

Then

$$a \in A \Longleftrightarrow \exists x\, P(x,a) .$$

But this x must be the tuple $\langle\langle n,a,0\rangle,\alpha\rangle \in M_a$, where α is the length of the computation $\{n\}(a) \simeq 0$, which gives the absoluteness property.

Now, let

$$a \in A \Longleftrightarrow \exists x \in M_a\, P(x,a) \Longleftrightarrow \exists x \in M\, P(x,a) ,$$

where P is $\Delta_0^{R_\Theta}$.

From the proof of Σ^*-collection we recall that there is a computable function f , defined on a whenever $\exists x \in M_a\ P(x,a)$, giving an a-index for a code for some x such that $P(x,a)$ holds. We also see from the proof of the selection theorem for numbers that f diverges if no such $x \in M_a$ exists. But then

$$A = \{a;\ f(a)\downarrow\}$$

which, by definition means that A is semicomputable. □

We call theorem 3.17 a (weak) normal form theorem. It is almost the same as 0,7 of Harrington [7], and the two theorems are easily derived from each other.

Note that this normal form theorem lies strictly between the two 'natural candidates':

$$\text{I} \quad a \in A \iff (\exists x \in sc(\Theta))\varphi(x,a)$$

$$\text{II} \quad a \in A \iff (\exists x \in \text{Full } sc(\Theta))\varphi(x,a) .$$

I does not for $k > 0$ include all Θ-semicomputable sets.
II may in general include more than the Θ-semicomputable sets.

Also note that in the formula P we only had natural numbers as parameters.

By $\Delta^*(R_\Theta)$-separation, we see that a subset A of I is $\Theta[a]$-computable if and only if it is $\Delta_1(R_\Theta)$ definable over M_a , without use of parameters.

<u>Corollary 3.18</u>

Let Θ_1 and Θ_2 be two type-k-theories. Then <u>i</u> and <u>ii</u> below are equivalent.

<u>i</u> $k+1-\text{en }\Theta_1 = k+1-\text{en }\Theta_2$ (Θ_1 and Θ_2 are equivalent theories).

ii R_{Θ_1} and R_{Θ_2} are Δ^* in each other, and for all $a \in I$, $M_a[\Theta_1] = M_a[\Theta_2]$.

Proof

i => ii. The envelope will always characterize the full section of a theory. Thus, for all $a \in I$

$$k+1-\text{sc}\ \Theta_1[a] = k+1-\text{sc}\ \Theta_2[a]$$

and thus $M_a[\Theta_1] = M_a[\Theta_2]$.

We prove that R_{Θ_1} is Δ^* in R_{Θ_2}. By theorem 3.17 Θ_1, as a set of tuples, will be Σ^* in R_{Θ_2}. Thus

$$\langle e,a,m\rangle \in \Theta_1 \iff \exists z \in M_a\ \varphi(\langle e,a,m\rangle,z)$$

for some Δ_0-in R_{Θ_2}-formula φ.

Let σ denote a tuple of the form $\langle e,a,m\rangle$. Then we have the following characterizations:

$$\begin{aligned}
\langle \sigma,\alpha\rangle \in R_{\Theta_1} \iff & (\exists x \in M_\sigma)(x \subseteq I \ \&\ x \text{ is a set of pairs} \\
& \&\ x \text{ is a wellfounded relation} \\
& \&\ (\forall b \in \text{dom } x)(\exists z \in M_{\sigma,b})\varphi(z,b) \\
& \&\ (\forall b \in \text{dom } x)(\forall c \leq_{\Theta_1} b)(\langle c,b\rangle \in x) \\
& \&\ a \in \text{dom } x \ \&\ |\sigma|_x = \alpha) \\
\iff & (\forall x \in M_\sigma)(x \subseteq I \ \&\ x \text{ is a set of pairs} \\
& \&\ (\forall b \in \text{dom } x)(\exists z \in M_{\sigma,b})\varphi(z,b) \\
& \&\ (\forall b \in \text{dom } x)(\forall c \leq_{\Theta_1} b)(\langle c,b\rangle \in x) \\
& \&\ |x| > \alpha \\
& \Rightarrow a \in \text{dom } x).
\end{aligned}$$

Again it is simple to verify the absoluteness properties. Note that comparing a prewellordering or a well-founded relation to an ordinal, is a Δ_1-operation over M_σ, uniform in σ.

To be able to write up an appropriate form of $(\forall a \subset \leq_{\Theta_1} b)(\langle c,b\rangle \in x)$, note that we need this particular formula only when b is a computation. But then $\{c;\ c \leq_{\Theta_1} b\}$ is uniformly computable in b.

ii => i

Let R_{Θ_1} and R_{Θ_2} be Δ^*, in each other. Then $\Sigma^*(R_{\Theta_1}) = \Sigma^*(R_{\Theta_2})$, and thus, by the theorem

$$k+1-\text{en}\ \Theta_1 = k+1-2\ \Theta_2 .$$

4. ABSTRACT APPROACH

In this part we make a few contributions to an "abstract" characterization of spectra. We introduce the notions of <u>abstract spectrum</u> and <u>nice family</u>, and show in particular that every abstract spectrum is the spectrum of some type-k theory Θ. We also prove some results on inductive definitions over these more general structures.

<u>Definition 4.1</u>

Let $\langle N_a \rangle_{a \in I}$ be a set of structures in V_I. We call $\langle N_a \rangle_{a \in I}$ a <u>family</u> if

<u>i</u> $I \in N_a$ for all $a \in I$.

<u>ii</u> $(\forall a,b \in I)(a \in N_b \iff N_a \subseteq N_b)$. □

The only thing we can derive from this definition is that

$$(\forall a \in I)(a \in N_a) .$$

For instance $\langle \{I,a\} \rangle_{a \in I}$ forms a family. But our $\mathrm{spec}(\Theta)$ also forms a family.

<u>Definition 4.2</u>

By an <u>abstract spectrum</u> we mean a family $\langle M_a \rangle_{a \in I}$ together with a relation $R \subseteq M: \bigcup_{a \in I} M_a$ such that

<u>i</u> <u>a</u> $\langle M_a \rangle_{a \in I}$ satisfies $\Sigma^*(R)$-collection.

<u>b</u> Each M_a is rudimentary closed in R.

<u>c</u> Each M_a is an abstract structure.

<u>ii</u> For all $a \in I$, $M_a = \Delta_a^*(R)$

where we use no parameters but natural numbers.

<u>iii</u> For all $a \in I$, $M_a \subseteq L^R_{\mathrm{rank}(M_a)}[I]$.

If $\langle\langle M_a\rangle_{a\in I}, R\rangle$ satisfies <u>i</u> only, we call $\langle M_a\rangle_{a\in I}$ a <u>nice family</u> relative to R . □

<u>Remark</u> Both <u>ii</u> and <u>iii</u> in definition 4.2 indicates that M_a will not be too wide. The following statement will be equivalent to <u>iii</u>.

<u>iii'</u> For all $a \in I$, $TC(M_a) = L^R_{\mathrm{rank}(M_a)}[I]$.

We introduced <u>iii</u> for technical reasons.

We will now study how inductive definitions behave inside nice families, so let $\langle M_a\rangle_{a\in I}$ be a nice family. We have observed that if each M_a is rudimentary closed, then each M_a satisfies Δ_0-separation. By Σ^*-collection and Δ_0-separation over M_a, we obtain Δ^*-separation over M_a .

We are going to illustrate the methods used to prove the main theorem of this section by one simple example.

<u>Lemma 4.3</u>

Let $\langle M_a\rangle_{a\in I}$ be a nice family.

Let $x \in M_a$ for some $a \in I$. Then rank $x \in M_a$.

<u>Proof</u>

First note that if $x \in M_a$, then x has a code A in M_a . It is trivial by a rudimentary function to find a code for $TC(x)$. Thus we may assume that x is transitive. Then

$$\begin{aligned}\mathrm{rank}\ x = \alpha \iff \exists f(&f \text{ is a function } \& \operatorname{dom} f = x\\ &\& f(\emptyset) = f(a) = 0 \text{ for all urelements } a \text{ in } x\\ &\& \forall y \in x\ (f(y) = \sup\{f(z)+1\ ;\ z\in y\}\\ &\& \alpha = \sup\{f(z)+1\ ;\ z\in x\}\ .\end{aligned}$$

$\iff \forall f$ (f is a function & dom $f = x$

& $f(\emptyset) = f(a) = 0$ for all urelements a in x.

& $\forall y \in x(f(y) = \sup\{f(z) + 1 \, ; \, z \in y\}$

$\Rightarrow \alpha = \sup\{f(z) + 1 \, ; \, z \in x\}$.

We know that the f above is unique; call it f_x. Using Σ^*-collection we prove by induction on rank x that rank $x \in M_a$ and $f_x \in M_a$.

We need the last claim to make the induction work for the first claim. □

The same method can be used to prove this general result.

Theorem 4.4

Let $\langle M_a \rangle_{a \in I}$ be a nice family. Let f be a function defined on $M = \bigcup_{a \in I} M_a$ such that when $x \in M_a$, then $f(x) \in M_a$. Assume also that the graph of f is Δ^*-definable. Define the function Γ inductively by

$$\Gamma(\gamma) = f(\bigcup_{\beta<\gamma} \Gamma(\beta)) .$$

Then, Γ is Δ^*-definable, and if $\gamma \in M_a$, then $\Gamma(\gamma) \in M_a$, and $\langle \Gamma(\beta) \rangle_{\beta<\gamma} \in M_a$.

The proof is almost identical to the proof for 4.3. First, give a Δ_1-definition for Γ. By induction on γ, verify that this definition is actually Δ^*. Use Σ^*-collection to make the induction work.

It is convenient to use a code in M_a for γ. Then for all $\beta < \gamma$, we uniformly find a b in that code which codes β. Σ^*-collection is simpler to use that way. □

Theorem 4.4. has one important corollary.

Corollary 4.5

Let $\langle M_a \rangle_{a \in I}$ be a nice family. Then the function

$$\Gamma(\gamma) = L_\gamma[I]$$

is Δ^*-definable, and if $\gamma \in M_a$, then $L_\gamma[I] \in M_a$.

Proof We will prove that the function

$$f(x) = \mathrm{DEF}(x)$$

satisfies the conditions in theorem 4.4. The proof is standard and we content ourselves with a brief sketch:

Give a natural Gödel-enumeration of the formulas $\{\varphi_e\}$. The set of Gödel-numbers will be a recursive set in ordinary recursion theory, and thus an element of M_a for all a.

A truth valuation is a function v such that

$$v(e,x_1,\dots,x_n) = \begin{cases} 0 & \text{if } \varphi_e(x_1,\dots,x_n) \text{ holds in } x \\ 1 & \text{if } \varphi_e(x_1,\dots,x_n) \text{ does not hold in } x. \end{cases}$$

We may prove by an induction on the rank of x that a truth-valuation exists in M_a whenever $x \in M_a$. The truth-valuation v is unique hence may be used to give a Δ^*-definition of $\mathrm{DEF}(x)$.

But since each M_a is rudimentary closed, we see that if $x \in M_a$ and $y \subseteq x$ is defined as

$$y = \{z \in x;\ x \models \varphi(z,\vec{x})\},$$

where $\vec{x}$ are parameters from x, then $y \in M_{a,\vec{x}}$. $M_{a,\vec{x}}$ denotes any M_b such that $a,\vec{x} \in M_b$.

Using a code for x and Σ^*-collection, we conclude that $\mathrm{DEF}(x) \in M_a$. $\square$

Theorem 4.6

Let F be a type $k+2$ functional. Let $\langle M_a\rangle_{a\in I}$ be a family nice relative to F. Then $k+1-sc(F,a) \subseteq M_a$ for all $a \in I$.

Remark D.B. McQueen [14] proved a theorem which essentially is a special case of this theorem.

Proof Let Γ be the operator defining recursion in F. We use theorem 4.4.to prove that $\{\langle\alpha,\Gamma_\alpha\rangle\}_{\alpha\in On}$ will be Δ^*-definable in F. By induction on the lengths of computations, we see that if $\langle e,\sigma,x\rangle$ is an F-computation, then $|\langle e,\sigma,x\rangle|_F \in M_\sigma$. Combining these results we obtain the theorem. Both claims are proved by analysing the operator Γ.

The case of the initial functions, $f(x,a) = x+1$, c_i, $ev^{(i)}$ and definition by cases is trivial. All computations have length one, and may be carried out in a Δ^* manner.

Composition $\{\langle n,e,e'\rangle\}(x) = \{e\}(\{e'\}(x))$ for some fixed $n \in \omega$. The length of computations is increased by one, and this is obviously Δ^*-definable. The treatment og Permutation/Evaluation and Iteration is similar to the treatment of composition.

Substitution in higher types: This and the next are the only non-trivial cases. We assume given a code e such that

$$\{e\}(e',x,a) = x(\lambda b\{e'\}(x,a,b))$$

whenever this makes sense, say b varies over $tp(i)$. Then

$$|\langle e,e',x,a\rangle|_F = \sup\{|\langle e',x,a,b\rangle|_F+1;\ b \in tp(i)\}$$

each $|\langle e',x,a,b\rangle|_F$ will be an element of $M_{x,a,b}$. By $\Sigma^*(F)$-

collection restricted to $tp(i)$, we see that $|\langle e,e',x,a\rangle|_F \in M_{x,a}$. The calculation will be

$$\{e\}(e',x,a) = z \iff \exists y \in tp(i)(\forall w \in tp(i-1) \\ (y(w) = \{e'\}(x,a,w) \ \& \ x(y) = z) .$$

This is Δ^* definable.

Recursion in F : This is quite analogous to the previous case. To obtain definability we make use of the fact that we work with definability relative to F . $\square$

Remark This theorem is analogous to the statement that recursion in a type-two F stops inside any F-admissible structure (except, of course, HF).

Theorem 4.7

Let $\langle\langle M_a\rangle_{a\in I}, R\rangle$ be an abstract spectrum. Then there is a type-k-theory Θ such that

i $\langle\langle M_a\rangle_{a\in I}, R_\Theta\rangle = \mathrm{Spec}(\Theta)$.

ii R and R_Θ are Δ^* in each other over $\langle M_a\rangle_{a\in I}$.

Proof It suffices to find a theory Θ such that $\Sigma^*(R) = k+1-\mathrm{en}(\Theta)$ over I . To this end we first establish three "claims":

Claim 1 $\Sigma^*(R)$ is normed.

Proof Let $A \subseteq I$ be a $\Sigma^*(R)$ set defined by

$$a \in A \iff \exists x \in M_a\ \varphi(a,x) \iff \exists x\ \varphi(a,x) .$$

If a and b are in A , we order them by letting $a < b$ if we find an x_a such that $\varphi(a,x_a)$ lower in the constructible hierarchy than any x_b such that $\varphi(b,x_b)$. Here we use clause iii in the definition of abstract spectra.

Our formal definitions will be

$$a < b \iff (\exists\alpha\in M_{\langle a,b\rangle})(\exists x\in L_\alpha^R[I])(\forall y\in L_\alpha^R[I])$$
$$(\varphi(a,x)\ \&\ \neg\varphi(b,y))$$

$$a \le b \iff (\exists\alpha\in M_{\langle a,b\rangle})(\exists x\in L_\alpha^R[I])(\varphi(x,b)\ \&$$
$$\forall\beta<\alpha\ ((\exists x\in L_\beta^R[I])\varphi(x,b) \Rightarrow$$
$$(\exists y\in L_\beta^R[i])\varphi(y,a)) .$$

These are both Σ^* . □

Claim 2 $\Sigma^*(R)$ is ω-parametrized.

Proof There is a Gödel-enumeration of all Δ_0-formulas φ_i without parameters, such that the relation

$$\Psi(i,a,\vec{x}) \iff \varphi_i(a,\vec{x})$$

will be Δ^*-definable. Then define $A\subseteq\omega\times I$ by

$$\langle i,a\rangle\in A \iff \exists x\in M_a\ \Psi(i,a,x) .$$

A is $\Sigma^*(R)$, and A is an ω-parametrization. □

Claim 3 $\Sigma^*(R)$ is closed under $\forall^I$ and $\exists^\omega$.

The first was remarked in section 3, the latter is trivial. ($M_{\langle a,n\rangle}=M_a$ for all $n\in\omega$). □

Now we define Θ to be the type-k-theory generated by $\Sigma^*(R)$. The ω-parametrization A will, by definition be Θ-semicomputable, and thus $\Sigma^*(R)\subseteq k+1-\mathrm{en}(\Theta)$. By claim 1 there is a norm on A , $\|\ \|_A$.

Let

$$f(i,a) = \begin{cases} 0 & \text{if } \langle i,a\rangle \in A \\ \text{undefined otherwise} \end{cases} .$$

This function we will code by $\langle 1,0\rangle$

Code functions

$$\begin{array}{lll} & C_i & \text{by } \langle 2,i\rangle \\ & ev^{(i)} & \text{by } \langle 3,i\rangle \\ & DC & \text{by } \langle 4,0\rangle \\ f(x,a) = & x+1 & \text{by } \langle 5,a\rangle \end{array}$$

$\langle 6,e,e'\rangle$ codes the composition of $\{e\}$ and $\{e'\}$

Permutation/Evaluation is coded by $\langle 7,0\rangle$

Iteration by $\langle 8,0\rangle$

Substitution in higher types by $\langle 9,0\rangle$.

We define the set of computations by induction on the ordinals by at each level α to close under the functions $\{\langle 2,i\rangle\}$ to $\{\langle 9,0\rangle\}$, and introduce the computations

$$\{\langle 1,0\rangle\}(\langle i,a\rangle) = 0$$

where $\langle i,a\rangle \in A$ and $\|\langle i,a\rangle\|_A = \alpha$.

Each initial segment of A is Δ^*-definable, and by the arguments from theorem 4.6 we see that $\{\langle\langle e,\sigma,x\rangle,\alpha\rangle;\ |e,\sigma,x|_\Theta = \alpha\}$ will be Δ^*-definable. But this is R_Θ . Thus $\Delta^*(R_\Theta) \subseteq \Delta^*(R)$, and $\Sigma^*(R_\Theta) \subseteq \Sigma^*(R)$. What is left is to prove that $R \in \Delta^*(R_\Theta)$. By clause <u>ii</u>, each element in M_a is $\Delta^*_a(R)$, and $\{\langle i,x,a\rangle;\ \varphi_i$ defines x over $M_a\}$ will be Δ^* . But then

$C = \{\langle i,b,a\rangle;\ \varphi_i$ defines a code A for a set x in M_a such that $x \in R$ and $b \in A\}$

is $\Sigma^*(R)$. This will be $\Sigma^*(R_\Theta)$.

The norm on C can be made so that for all i,a we have

$$(\forall b_1,b_2)\langle i,b_1,a\rangle \in C \;\&\; \langle i,b_2,a\rangle \in C \Rightarrow \|\langle i,b_1,a\rangle\|_C = \|\langle i,b_2,a\rangle\|_C.$$

The following will then be a Σ^*-definition of R from R_Θ :

$$R = \{x;\ (\exists c,i,a\ \exists A)(c \in C \;\&\; A = \{b;\ \|\langle i,b,a\rangle\|_C < \|c\|_C\} \;\&\; A \text{ codes } x\}.$$

Since we may do the same with $M \setminus R$, the proof is complete.
□

5. PARTIAL CONSTRUCTIBILITY AND NICE FAMILIES

When we compute relative to ${}^{k+2}E$ it may happen that we perform a number of computations say one for each $a \in I$, such that the computations are uniform in this a, and then one "big" computation at the end. We would like to do the same with constructibility, i.e. construct a set by giving relative constructions of all its elements in a uniform way. Our intuition will be that information about I is given as uniform information, not spesific information about each element of I.

Therefore, we want a notion of constructibility that makes a set constructible over a set of utelements, without making all urelements constructible.

To motivate our definition, observe the following result:

Lemma 5.1

Let $A \subseteq V_I$, $\alpha \in On$. Let $x \in L_\alpha^A[I]$. Then there are $a_1,\dots,a_n \in I$, $\alpha_1,\dots,\alpha_m \in On$ such that

x is Δ_0-definable in the parameters $a_1,\dots,a_n$ and $L_{\alpha_1}^A[I],\dots,L_{\alpha_m}^A[I]$.

$\square$

We have not found a suitable reference, so we indicate the proof:

Each element in L_α is definable by some parameters in $L_{\beta_1},\dots,L_{\beta_n}$ for some $\beta_1,\dots,\beta_n < \alpha$.

These parameters are again definable by other parameters. By well-foundedness we see that after a finite number of steps we obtain parameters defined over some L_β using no parameters from L_β. What is left now is a technical argument that 'replaces' a parameter in a formula by a definition of this parameter. We omit these details.

In the following definition we think of P as a nice set of ordinals, for instance the lengths of computations with natural number arguments. (This corresponds to the ordinals called <u>constructible</u> in Harrington [7]) . We give, however, the definition in a general form.

<u>Definition 5.2</u>

Let $A \subseteq V_I$, $\alpha \in On$, $P \subseteq On$.

We say that

x is constructible in A on level α partially relative to P

if

<u>i</u> $x \in L_\alpha^A[I]$.

<u>ii</u> x is Δ_0-definable in parameters $L_{\alpha_1}^A[I],\dots,L_{\alpha_n}^A[I]$ where $\alpha_1,\dots,\alpha_n$ are elements in $P \cap \alpha$.

These sets we denote by $S_\alpha^A(P)$. $\square$

It is trivially seen that, if $\alpha > \beta$ and P contains no ordinals between α and β , then $S_\alpha^A(P_a) = S_\beta^A(P_a)$.

<u>Lemma 5.3</u>

Let Θ be a theory and $\langle M_a\rangle_{a\in I} = \mathrm{Spec}(\Theta)$. Let $P \subseteq M_a \cap O_n$ be $\Delta_1^{R_\Theta}$-definable over M_a . Then

$$S_\alpha^{R_\Theta}(P) \in M_a \quad \text{for all} \quad \alpha \in M_a \ .$$

Moreover, $S_\alpha^{R_\Theta}(P)$ is M_a-clean.

<u>Proof</u> M_a satisfies Δ_1-separation, so $P \cap \alpha \in M_a$. We also

have that $P \cap \alpha$ is clean. By 4.5, $L_\alpha^{R_\Theta}[I] \in M_a$. The condition "$\Delta_0$-definable" is Δ_1-definable as may be seen from the proof of corollary 4.5. Thus $S_\alpha^{R_\Theta}(P) \in M_a$ by Δ_1-separation. Moreover, all Δ_0-definable elements of $TC(M_a)$ will by Δ_0-separation be elements of M_a.

All elements of $S_\alpha^{R_\Theta}(P)$ are Δ_0-definable in parameters from M_a, since $P \subseteq M_a$. Thus $S_\alpha^{R_\Theta}$ is clean. $\square$

<u>Corollary 5.4</u>

Let $P_a = \{\alpha;\ \langle\langle n,a,\vec{m}\rangle,\alpha\rangle \in R_\Theta$ for some n and $\vec{m}$ from $\omega\}$.

Let $\alpha_a = \sup\{On \cap M_a\}$.

Then

$$M_a = S_{\alpha_a}^{R_\Theta}(P_a) .$$

<u>Proof</u>

P_a is $\Delta_1^{R_\Theta}$-definable over M_a, and $P_a \subseteq M_a$, so $\supseteq$ follows from lemma 5.3.

All codes A for sets in M_a will be of the form

$$\{b;\ (\exists\alpha' < \alpha)(\langle\langle n_0,a,b,0\rangle,\alpha'\rangle \in R_\Theta\} \text{ for some } \alpha \in P_a .$$

The reason for this is that if the characteristic function of A is $\lambda b\{n_0\}(a,b)$, then we may find a computation in a of length α greater than the lengths of the computations $\{n_0\}(a,b)$. But then

$$A \in S_\alpha^{R_\Theta} .$$

Let A code x. By a trivial argument we see that

$$x \in L_{\alpha+rn(x)}^{R_\Theta}[I]$$

and x is definable from A and $L_{\alpha+rn(x)}^{R_\Theta}[I]$.

For any $\beta \geq \alpha + rn(x)$ we will have x definable from A and $L_\beta^{R_\circledR}[I]$, and there will always be some $\beta \in P_a$ of this kind. But then

$$x \in S_\beta^{R_\circledR}(P_a) \ .$$

$\square$

We might as well have defined M_a in this way and obtained a definition parallel to the one of the companion (see the introduction). But that would be a bit ad hoc.

Also note that in the definition of P_a we jump over a lot of ordinals from M_a . We may have $\alpha \in M_a$, $\beta < \alpha$, $\beta \in M_a$ but $\beta \notin S_\alpha^{R_\circledR}(P_a)$. This is essential since $\alpha \cap M_a$ does not belong to M_a for all $\alpha \in M_a$.

<u>Definition 5.5</u>

Let $\langle M_a \rangle_{a \in I}$ be a family nice relative to some R . We define by induction on α two concepts:

<u>i</u> The family $\langle M_a^\alpha(R) \rangle_{a \in I}$.

<u>ii</u> α is <u>a-necessary</u> .

We always denote by P_a^α the set of a-necessary ordinals less than α .

1. $\langle M_a^0(R) \rangle_{a \in I} = \langle a, I \rangle_{a \in I}$

 0 is a-necessary.

2. $M_a^\alpha(R) = S_\alpha^R(P_a^\alpha)$ (See definition 5.2).

 α is a-necessary if

 <u>i</u> $\alpha = \beta + 1$ for some a-necessary β .

 <u>ii</u> There is a Δ_0-formula φ with parameters from $M_a^\alpha(R)$ such that

$$\forall b \in I \; \exists \gamma \in M^{\alpha}_{\langle a,b\rangle}(R) \, \varphi(a,b,\gamma)$$

and

$$\forall \gamma \in M^{\alpha}_{a} \; \exists b \in I \; (\forall \delta \in M^{\alpha}_{\langle a,b\rangle}(R))(\varphi(a,b,\delta) \Rightarrow \delta > \gamma) \quad \Box$$

This definition should be understood from the following reflections: In the proof of theorem 4.6 we found the length of a computation in a, either by adding 1 to another length of a computation, or, by using Σ^*-collection, to find an upper bound for a set of uniform computations. By the definition above, we see that the ordinals of a-necessity are obtained in a parallel way, and that the families

$$\langle M^{\alpha}_{a}(R)\rangle_{a\in I}$$

will 'converge' to the least nice family relative to R. Obviously all a-necessary ordinals will be in M_a. Let $N_a = \bigcup_{\alpha\in On} M^{\alpha}_{a}(R)$. We will verify that $\langle N_a\rangle_{a\in I}$ is a nice family relative to R.

Both Σ^*-collection and rudimentary closure are trivial by the definition. To see that $M^{\alpha}_{a}(R)$ is an abstract structure, we use a natural notation system for the a-necessary ordinals:

1 is an a-code for 0.

If x is an a-code for α, then 2^x will be an a-code for $\alpha + 1$.

If α is a-necessary by clause <u>ii</u> in the definition, there will be a Δ_0-formula φ with parameters from $M^{\alpha}_{a}(R)$ such that

$$\forall b \in I \; \exists \gamma \in M^{\alpha}_{\langle a,b\rangle}(R) \, \varphi(a,b,\gamma).$$

and these γ's have to be chosen cofinal in α. We may assume that the parameters from $M^{\alpha}_{a}(R)$ are all of the kind $x_i = L^{R}_{\alpha_i}[I]$, where α_i are a-necessary ordinals.

Also assume that φ has Gödel-number j and $\alpha_1,\ldots\alpha_n$ has a-codes $e_1,\ldots,e_n$. Then $3^{\langle e_1,\ldots,e_n\rangle}\cdot 5^j$ will be an a-code for α.

This ends the construction of the notation system.

The relation

"e is an a-notation for α" will be Δ^*.

Now let $x \in M_a$ be defined by formula φ_i from parameters $L^R_{\alpha_1},\ldots,L^R_{\alpha_n}$. Assume $\alpha_1,\ldots,\alpha_n$ have a-notations $e_1,\ldots,e_n$. Then we associate to x the number

$$\langle i,e_1,\ldots,e_n,a\rangle .$$

Using this it is not hard, in a Δ^*-manner, to define a code for x.

□

If we simulated the definition of M^α_a in the ω-case, we would stop at the first R-admissible ordinal.

Also note that given R, we use theorem 4.7 and see that R induces a least type-k-theory on I by this definition.

6. CHARACTERIZING ENVELOPES

In this section we characterize those type-k- theories Θ such that $k+1-\mathrm{en}(\Theta)$ is the $k+1$-envelope of some normal typ k+2-functional F.

The starting point is the following result over ω: A finite theory Θ over ω is the theory of a normal type-two functional if and only if α is not R_Θ-recursively Mahlo (see theorem G of the introduction).

As a preparation for the general result we give a brief sketch of the ω-result:

Let $f\colon On \to On$ be a normal function Δ_1 in R_Θ and without R_Θ-admissible fix-points. Define inductively

$$R^*_0 = \emptyset$$

$$R^*_\lambda = \bigcup_{\gamma<\lambda} R^*_\gamma \text{, when } \lambda \text{ is a limit}$$

$$R^*_{\gamma+1} = R^*_\gamma \cup \{\langle\langle n,m\rangle, \mathrm{rank}(R^*_\gamma)\rangle;\ m \text{ is a } \Theta\text{-computation of length } < f(\mathrm{rank}(R^*_\gamma)+1) \text{ and } n \text{ is a subcomputation of } m\}$$

$$\cup \ \{\langle 0, f(\mathrm{rank}(R^*_\gamma)+1)\rangle\}\ .$$

Let $R^*_\Theta = \bigcup_{\gamma<\alpha} R^*_\gamma$.

It is easily verified that R^*_Θ is Δ_1 in R_Θ, and that R_Θ at any level of R^*_Θ is definable from R^*_Θ. Thus an R^*_Θ-admissible ordinal must be an R_Θ-admissible ordinal.

At limit stages the rank of R^*_Θ is a fix-point for f. Between these ordinals, we have put in prewellorderings of suitable lengths to guarantee that we do not find any R^*_Θ-admissible ordinal there. Thus α is forced to be the least R^*_Θ-admissible ordinal, and $L_\alpha^{R^*_\Theta} = L_\alpha^{R_\Theta}$.

R^*_Θ can be coded as a type-2-functional F . Due to the minimality of α , the 1-section of F is equal to $L_\alpha^{R^*_\Theta} \cap 2^\omega = L_\alpha^{R_\Theta} \cap 2^\omega$. Since R^*_Θ and R_Θ are Δ_1 in each other we may from this conclude that the 1-envelope of F consists of exactly the $\Sigma_1(R_\Theta)$ subsets of ω , which by the imbedding theorem is the 1-envelope of Θ .

On the other hand, if $1\text{-en}(\Theta) = 1\text{-en}(F)$ for some normal type two F , the set of computations in F will be Σ_1-definable in R_Θ . Moreover, if Θ' is the theory of F , $R_{\Theta'}$ will be $\Delta_1(R_\Theta)$. α will be the least $R_{\Theta'}$-admissible ordinal. Define $f(\gamma)$ = least β such that $R_{\Theta'} \cap \omega \times \gamma$ is Δ_1-definable in $\langle L_\beta^{R_\Theta}, R_\Theta\rangle$ by a given Δ_1-definition of $R_{\Theta'}$. This f will be a counterexample to Mahlo-ness.

We are going to copy this proof. Our problem will be to find a suitable generalization of Mahlo and admissibility.

In the proof above we jumped over all admissibles, since recursion stops at admissible ordinals. Recalling theorem 4.6, we remember that nice families are closed under recursion.

The concept of nice family is not a direct generalization of admissibility, but of admissibility pluss local countability. Local countability corresponds to the condition that each M_a is an abstract structure.

<u>Theorem 6.1</u>

Let Θ be a type-k-theory.
Let Spec $\Theta = \langle\langle M_a\rangle_{a\in I}, R_\Theta\rangle$, and let

$$\alpha = \sup(\mathrm{On} \cap M_a ; a \in I) .$$

Then the following statements are equivalent:

1. $k+1-\mathrm{en}\ \Theta = k+1-\mathrm{en}\ F$, for some normal type k+2-functional F.

2. There is a $R \subseteq M = \bigcup_{a\in I} M_a$ such that

   i R is Δ^*-definable in R_Θ.

   ii R_Θ is Δ^*-definable in R.

   iii $\langle M_a^\alpha(R)\rangle_{a\in I} = \langle M_a\rangle_{a\in I}$.

Remark Recall from definition 5.5 that $\langle M_a^\alpha(R)\rangle_{a\in I}$ will be the least family nice relative to R.

Note that we do not have to bother with condition ii, since given R' satisfying i and iii, the direct union of R_Θ and R would satisfy i, ii and iii.

Proof

1 => 2

Assume $k+1-\mathrm{en}(\Theta) = k+1-\mathrm{en}(F)$.

Let $R_F = R_{\Theta'}$, where Θ' is the theory of F. Then

$$\langle M_a[F]\rangle_{a\in I} = \langle M_a[\Theta]\rangle_{a\in I}\text{, and, by}$$

corollary 3.18

$$R_F \in \Delta^*(R_\Theta)\ .$$

It is not hard to see from the proof of theorem 4.7 that

$$\langle M_a^\alpha(R_F)\rangle_{a\in I} = \langle M_a[F]\rangle_{a\in I}\ .$$

R_F will fulfill the conditions on R in 2, since

$$\langle M_a^\alpha(R_F)\rangle_{a\in I} = \langle M_a[F]\rangle_{a\in I} = \langle M_a[\Theta]\rangle_{a\in I} = \langle M_a\rangle_{a\in I}\ .$$

2 => 1

Let R be given as in the theorem.

Define the functional F by

$$F(A) = \begin{cases} 1 & \text{if } A \text{ is a code for an element in } R \\ 0 & \text{otherwise} \end{cases}$$

Claim 1

$F \cap M$ is $\Delta^*(R_\Theta)$-definable.

Proof

$$F(A) = 1 \iff (\exists x)(A \text{ is a code for } x \text{ and } x \in R)$$
$$\iff (\forall x)(A \text{ is a code for } x \Rightarrow x \in R)$$

$$F(A) = 0 \iff A \text{ is not a code or } (\exists x)(A \text{ is a code for } x \text{ and } x \notin R)$$
$$\iff A \text{ is not a code or } (\forall x)(A \text{ is a code for } x \Rightarrow x \notin R) .$$

It is easily seen that this definition is absolute, since each M_a is an abstract structure. □

Claim 2

R is $\Delta^*(F)$-definable.

Proof

$$x \in R \iff (\exists A)(A \text{ is a code for } x \text{ and } F(A) = 1)$$
$$\iff (\forall A)(A \text{ is a code for } x \Rightarrow F(A) = 1) .$$

Again we use the fact that each M_a is an abstract structure to prove absoluteness. □

Claim 3

The definitions in claims 1 and 2 hold when restricted to any nice family $\langle N_a \rangle_{a \in I}$.

Proof

We only used that

$$\text{'}A \text{ is a code for } x\text{'}$$

is Δ_1-definable over any rudimentary closed abstract structure, and that each N_a is an abstract structure. $\square$

We know that $\langle M_a(F, {}^{k+2}E)\rangle_{a\in I}$ is a family nice relative to F. Thus $\langle M_a(F, {}^{k+2}E)\rangle_{a\in I}$ is a family nice relative to R.

Then

$$\langle M_a^{\alpha}(R)\rangle_{a\in I} \subseteq \langle M_a(F, {}^{k+2}E)\rangle_{a\in I} \subseteq \langle M_a\rangle_{a\in I} \cdot$$

since $\langle M_a^{\alpha}(R)\rangle_{a\in I}$ is the minimal family relative to R. Inclusion between families means inclusion in each component.

But

$$\langle M_a^{\alpha}(R)\rangle_{a\in I} = \langle M_a\rangle_{a\in I} .$$

So $\mathrm{Spec}(F, {}^{k+2}E)$ is the same as $\mathrm{Spec}(\Theta)$.

$R_{F, {}^{k+2}E}$ is $\Delta^*(F)$-definable, so

$$R_{F, {}^{k+2}E} \in \Delta^*(R_\Theta) .$$

On the other hand

$$F \cap M \text{ is } \Delta^*(R_F)\text{-definable.}$$

Then $R_{F, {}^{k+2}E}$ and R_Θ are Δ^* in each other, and by corollary 3.18

$$k+1-\mathrm{en}\ F, {}^{k+2}E = k+1-\mathrm{en}\ \Theta .$$

$\square$

We promised a generalization of the characterization result over ω. Recall that an ordinal α is R-recursively Mahlo if all R-recursive functions are closed in some R-admissible $\beta < \alpha$.

It is this formulation of the Mahlo property we will generalize.

Definition 6.2

a Let $\langle M_a \rangle_{a \in I}$ be a family, $M = \bigcup_{a \in I} M_a$ and

$$f : M \to M \text{ a function.}$$

We say that f is closed in $\langle M_a \rangle_{a \in I}$ if

$$(\forall a \in I)(x \in M_a \Rightarrow f(x) \in M_a) .$$

b Let $\langle M_a \rangle_{a \in I}$ be a family nice relative to some $R \subseteq M$. We say that $\langle M_a \rangle_{a \in I}$ is R-impenetrable if for all normal functions

$$f : On \cap M \to On \cap M$$

such that f is $\Delta^*(R)$-definable and closed in $\langle M_a \rangle_{a \in I}$, there is a family

$$\langle N_a \rangle_{a \in I} \subsetneqq \langle M_a \rangle_{a \in I} ,$$

nice relative to R such that f is closed in $\langle N_a \rangle_{a \in I}$.

c We call f rapid if f is a counterexample to impenetrability.

□

Note that over ω, impenetrability and Mahloness coincide. In that case there is only one M_a, and niceness corresponds to admissibility.

Theorem 6.3

Let Θ be a type-k-theory. Then the following statements are equivalent:

1. $k+1-\mathrm{en}(\Theta) = k+1-\mathrm{en}(F)$, for some normal type-k+2-functional F.

2. $\langle M_a[\Theta] \rangle_{a \in I}$ is R_Θ-penetrable.

Remark In the ω-case we gave a direct proof of this theorem. Here we have sorted out one half of the argument in theorem 6.1, i.e. the construction of F from R and vice versa. Here we will give the general form of the other, namely the construction of R from a rapid function f and vice versa. Thus we use theorem 6.1.

Proof

1 => 2 Let $R \in \Delta^*(R_\Theta)$ be such that

$$\langle M_a^\alpha(R)\rangle_{a\in I} = \langle M_a\rangle_{a\in I}$$

i.e. $\langle M_a\rangle_{a\in I}$ is the least nice family relative to R . (From the proof of theorem 6.1 we know that R_F is a suitable representative for R) .

We recall from section 5 that P_a denotes the lengths of $\Theta[a]$-computations, and that $S_\alpha^{R_\Theta}(P_a)=M_a$. Also $\langle S_\alpha^{R_\Theta}(P_a)\rangle_{a\in I} = \langle M_a^\alpha(R)\rangle_{a\in I}$.

But R will be Δ^*-definable over $\langle\langle S_\alpha^{R_\Theta}(P_a)\rangle_{a\in I}, R_\Theta\rangle$ and each 'finite part' of R will be Δ^*-definable over $\langle\langle S_\beta^{R_\Theta}(P_a)\rangle_{a\in I}, R_\Theta\rangle$ for some $\beta < \alpha$.

Now let γ be given.

We let $f(\gamma+1)$ be the least β such that

$$R \cap (\bigcup_{a\in I} S_{\gamma+1}^{R_\Theta}(P_a))$$

is defined inside $\langle S_\beta^{R_\Theta}(P_a)\rangle_{a\in I}$. This will be made more precise later.

Note that

$$\bigcup_{a\in I} S_{\gamma+1}^{R_\Theta}(P_a) = L_{\gamma+1}^{R_\Theta}[I]$$

which is $\Delta^*(R_\Theta)$-definable. Thus

$$x = R \cap L_{\gamma+1}^{R_\Theta}[I]$$

is $\Delta^*(R_\Theta)$-definable as a function of γ, and there are $\Delta_0(R_\Theta)$-formulas φ and ψ such that

$$\begin{aligned} x = R \cap L_{\gamma+1}^{R_\Theta}[I] &\iff \exists y\ \varphi(x,y,\gamma) \\ &\iff \forall y\ \psi(x,y,\gamma) \end{aligned}$$

where the formulas are absolute with respect to all M_a having γ as an element.

Then we can give a precise definition of f at successor stages:

$f(\gamma+1) = \beta \iff \beta$ is the least ordinal $\geq \gamma+1$ such that

$$\begin{aligned} &(\exists x \in L_\beta^{R_\Theta})(\langle S_\beta^{R_\Theta}(P_a)\rangle_{a\in I} \models \exists y\ \varphi(x,y,\gamma) \\ &\quad \& \ \langle S_\beta^{R_\Theta}(P_a)\rangle_{a\in I} \models \forall y\ \psi(x,y,\gamma)) . \end{aligned}$$

By $\Sigma^*(R_\Theta)$-collection we see that $f(\gamma+1) \in M_a$.

We define f to be continuous at limit stages.

We will prove that f will be rapid, i.e. that $\langle M_a\rangle_{a\in I}$ is not R_Θ-impenetrable. To obtain a contradiction, assume that f is closed in

$$\langle N_a\rangle_{a\in I} \subsetneqq \langle M_a\rangle_{a\in I}$$

and that $\langle N_a\rangle_{a\in I}$ is nice relative to R_Θ.

Now let $\gamma \in N_b$. Then $\beta = f(\gamma+1) \in N_b$, and $\langle S_\beta^{R_\Theta}(P_a)\rangle_{a\in I} \in N_b$.

Our Δ^*-definition of $R \cap L_{\gamma+1}^{R_\Theta}[I]$ is valid inside $\langle S_\beta^{R_\Theta}(P_a)\rangle_{a\in I}$ by definition of f. Thus

$$R \cap \bigcup_{a\in I} N_a$$

will be Δ^*-definable over $\langle N_a \rangle_{a \in I}$ and R_{Θ} , so $\langle N_a \rangle_{a \in I}$ is nice relative to R . Then $\langle M_a^{\alpha}(R) \rangle_{a \in I}$, which is the minimal family nice relative to R , will be included in $\langle N_a \rangle_{a \in I}$. But, by assumption

$$\langle N_a \rangle_{a \in I} \subsetneqq \langle M_a \rangle_{a \in I} = \langle M_a^{\alpha}(R) \rangle_{a \in I} .$$

This is a contradiction.

2 => 1 Notation: Let $\beta \in On$. By $R_{\Theta} \restriction \beta$ we mean

$$R_{\Theta} \restriction \beta = \{a;\ (\exists \gamma \in \beta)(\langle a, \gamma \rangle \in R_{\Theta}) ,$$

i.e. the set of Θ-computations of length less than β .

Let f be rapid.

Let $R_{\gamma} = \{\langle a,b,\gamma \rangle;\ a \in R_{\Theta} \restriction f(\gamma)\ \&\ b \in R_{\Theta} \restriction f(\gamma)\ \&\ a \leq_{\Theta} b\}$.

Let $R = \bigcup_{\gamma < \alpha} R_{\gamma}$.

Obviously R is $\Delta^*(R_{\Theta})$-definable, since f is . R_{Θ} will be $\Delta^*(R)$-definable over any R-nice family, since R is just a 'collapsing' of R_{Θ} . Let

$$\langle N_a \rangle_{a \in I} = \langle M_a^{\alpha}(R) \rangle_{a \in I} \text{ , i.e. the least R-nice family.}$$

We claim that f is closed in $\langle N_a \rangle_{a \in I}$.

Proof

Let $\gamma \in N_a$, and suppose $\gamma \in M_a^{\beta}$ for some a-necessary β . Then

$$\{\langle a,b \rangle\ ,\ \langle a,b,\gamma \rangle \in R\}$$

is a prewellordering of length $f(\gamma)$. But then $f(\gamma)$ is an ordinal in N_a , since N_a is an abstract structure, and the claim is proved.

Since f is rapid, f is not closed in any proper subfamily of $\langle M_a \rangle_{a \in I}$. Then $\langle M_a^\alpha(R) \rangle = \langle M_a \rangle_{a \in I}$, and we may use theorem 6.1.

$\square$

7 RECURSION IN THE SUPERJUMP

The superjump S^3 was originally introduced by R. Gandy [6]. It can be generalized to a functional S^{k+3} of type $k+3$ defined as follows:

$$S^{k+3}(e,F) = \begin{cases} 0 & \text{if } \{e\}^F\downarrow \\ 1 & \text{if } \{e\}^F\uparrow \end{cases}$$

where e is a Kleene-index and F a type-k+2 functional. S^{k+3} was intensively studied by Harrington in his thesis [7]. Some of his main conclusions were:

i Recursion in S^{k+3},F fails to be a theory on one important point:

A set may be semirecursive and co-semirecursive without being recursive.

ii Given $F \in tp(k+2)$, there is a theory Θ such that

$$k+1-sc\ \Theta[a] = k+1-sc\ (S^{k+3},F,a) \quad \text{for all } a \in I$$

$$k+1-en\ \Theta \subseteq k+1-en\ (S^{k+3},F)\ .$$

The theory in ii is obtained from a hierarchy generalizing the hierarchy introduced by Harrington in [8]. We call this theory the Harrington-theory for F,S^{k+3}.

The starting point for the investigations in this section was the main results of Harrington [8]:

a Let $\rho^F = \omega_1^{S^3,F}$ be the ordinal for recursion in S^3,F. Then ρ^F is the first ordinal which is recursively Mahlo in F.

b $L^F_{\rho^F} \cap \mathcal{P}(\omega) = 1-sc(S^3,F)$.

We will characterize those theories that are the Harrington-theories of S^{k+3},F for some $F \in tp(k+2)$. This is done much in the flavor of our characterization of theories in normal F's.

We define a notion of strongly impenetrable, which in the ω-case coincides with impenetrability and Mahloness, but which in the higher type case is probably a strictly stronger notion.

We prove that Harrington-recursion in S^{k+3},F gives us a strongly impenetrable structure, and that his hierarchy for S^{k+3},F will close up inside such a structure. Then if a family may be 'collapsed' to a minimal strongly impenetrable family, this family will be the spectrum of Harrington-recursion in S^{k+3} and some type-$k+2$ F.

Now we recall the definition of Harrington's hierarchy for recursion in S^{k+3},F. (See definition 3.5 of Harrington [7]).

<u>Definition 7.1</u>

<u>a</u> For any $X \subseteq tp(k)$ and $e \in \omega$, let W_e^X be the set

$$W_e^X = \{x;\ \{e\}_\mu^X(x) \simeq 0\},$$

where $\{e\}_\mu^X$ is the e^{th} type k+1 function arithmetic (i.e. μ-recursive) in ${}^{k+2}E$ and X.

<u>b</u> We will now define:

<u>i</u> A subset $\mathcal{N}^F$ of I

<u>ii</u> A function $|\ |^F : \mathcal{N}^F \to$ Ordinals; and for each ordinal σ in the range of $|\ |^F$;

<u>iii</u> A subset, H_σ^F of I.

This will be done by simultaneously, for each $a \in I$ inductively defining a set of integers, $\mathcal{N}_a^F$, and a function $|\ |_a^F : \mathcal{N}_a^F \to$ Ordinals.

$\mathcal{N}^F$, $|\ |^F$ and H^F_σ will then at the same time be defined by:

$$\mathcal{N}^F = \{\langle m,a\rangle;\ m \in \mathcal{N}^F_a,\ a \in I\}$$

$$|\langle m,a\rangle|^F = |m|^F_a$$

$$H^F_0 = \emptyset$$

$$H^F_{\sigma+1} = \{\langle e,a,0\rangle;\ a \in W_e^{H^F_\sigma}\}$$

$$\cup\ \{\langle e,a,x+1\rangle;\ F(\{e\}_\mu^{\langle H^F_\sigma,a\rangle}) = x\}$$

and

$$H^F_\lambda = \{\langle b,c\rangle;\ b \in \mathcal{N}^F,\ |b|^F < \lambda,\ c \in H^F_{|b|^F}\} .$$

(For the sake of convenience, all superscripts will from now on be omitted whenever possible).

(i) $1 \in \mathcal{N}_a$, $|1|_a = 0$

(ii) $x \in \mathcal{N}_a \Rightarrow 2^x \in \mathcal{N}_a$ and $|2^x|_a = |x|_a + 1$

(iii) given $m,e \in \omega$, if $m \in \mathcal{N}_a$, $|m|_a = \sigma$,

and if

$$W_e^{\langle H_\sigma,a\rangle} \subseteq \mathcal{N},$$

then $3^m\cdot5^e \in \mathcal{N}_a$, $|3^m\cdot5^e|_a$ = the 1st limit ordinal greater than σ and greater than $|b|$ for all

$$b \in W_e^{\langle H_\sigma,a\rangle} .$$

(iv) given $e \in \omega$, if there is a limit ordinal λ such that

(a) $\forall m,e' \in \omega\ \forall b \in I\ \ \lambda \neq |3^m\cdot5^{e'}|_b$

(b) $\forall b \in \mathcal{N},\ |b| < \lambda \Rightarrow e \in \mathcal{N}_{\langle a,b\rangle}$ and

$|e|_{\langle a,b\rangle} < \lambda$

then $7^e \in \mathcal{N}_a$, $|7^e|_a$ = the 1st limit ordinal satisfying (a) and (b).

For $a \in I$ and $m \notin \mathcal{N}_a$, let $|m|_a = |\langle m,a\rangle| = \infty$. $\square$

<u>Remark</u> This definiton is almost identical to Harrington's. The only difference is that we have used "μ-recursive in ${}^{k+2}E,X$" where Harrington uses "primitive recursive in ${}^{k+2}E,X$". The stronger condition seems to be necessary in order to prove the equivalence to Kleene recursion.

A recursion theory is defined from this hierarchy in the following way:

For $e \in \omega$ and $a \in I$, let

$\{e\}^{F,S^{k+3}}(a) \simeq x$ mean that

$e = \langle e_0,e_1\rangle$, $e_0 \in \mathcal{N}_a$ and

$\{e_1\}_\mu^{H_{|e_0|_a}} \simeq x$.

It is this theory that is the Harrington-theory for F,S^{k+3} and which has all the nice properties listed above.

<u>Lemma 7.2</u>

Let Θ be a theory, let $\langle M_a\rangle_{a\in I} = \mathrm{Spec}(\Theta)$ and let F be Δ^*-definable over $\langle\langle M_a\rangle_{a\in I},R_\Theta\rangle$. Then the following set is Δ^*-definable over $\langle\langle M_a\rangle_{a\in I},R_\Theta\rangle$:

$\{\langle H_\sigma^F,a,e,\sigma\rangle;$ e is an a-notation for $\sigma\} \cap M$ $(= \bigcup_{a\in I} M_a)$,

where e is an a-notation for σ if $e \in \mathcal{N}_a$ and $|e|_a^F = \sigma$.

<u>Proof</u> By the following claim, the lemma will follow by theorem 4.4:

<u>Claim</u>: Let $\gamma \in On$, and assume

$A = \{\langle H_\sigma^F,a,e,\sigma\rangle;$ e is an a-notation for σ and $\sigma < \gamma\}$ is given.

We may then, in a Δ^*-way recognize the notations for γ, and, by a Δ^*-definition define H^F_γ.

Proof

a Definition of H^F_λ when λ is a limit and $\lambda = \gamma$.
By the original definition of H^F_λ, we see

$$\langle b,c\rangle \in H^F_\lambda \iff (\exists\langle H^F_\sigma, a, e, \sigma\rangle \in A)(c \in H^F_\sigma \;\&\; (b)_0 = e \;\&\; (b)_1 = a) .$$

b Assume $\gamma = \lambda$ is a limit.
Answer to the problem: Is $3^m \cdot 5^e$ an a-notation for λ? :

$3^m \cdot 5^e$ is an a-notation for λ if $m \in \mathcal{N}_a$ at some level before λ, say $|m|_a = \sigma$, i.e. $\langle H^F_\sigma, a, m, \sigma\rangle \in A$, and if $W_e^{\langle H_\sigma, a\rangle} \subseteq \mathcal{N}$ and is unbounded in the natural ordering of the notations up to level λ.

Given a, m, e it is simple to find σ and H_σ fom A, if they exist. Note that the arithmetic hierarchy for H^F_σ, ${}^{k+2}E$ is defined by a simple inductive definition of length ω, and will when we are inside some M_b, be contained in M_b. Thus we may construct $W_e^{\langle H_\sigma, a\rangle}$ from H_σ and a. Now it is simple to check if $W_e^{\langle H_\sigma, a\rangle}$ has the required properties.

c Assume $\gamma = \lambda$ is a limit.
How to answer the problem: Is 7^e an a-notation for λ? :

First, as in b, check that λ is not given any notation of the type $|3^m, 5^{e'}|_b$. If not, see if for any $b \in \mathcal{N}_\lambda$, e will be an element of $\mathcal{N}_{\langle a,b\rangle}$ and $|e|_{\langle a,b\rangle} < \lambda$, and that no $\lambda' < \lambda$ has these properties. This can be done in a Δ_0-manner.

d Assume $\gamma = \gamma_0 + 1$. We define H^F_γ from $H^F_{\gamma_0}$:

$$H^F_\gamma = \{\langle e,a,0\rangle \; ; \; a \in W_e^{H^F_{\gamma_0}}\}$$

$$\cup \{\langle e,a,x+1\rangle \; ; \; F(\{e\}_\mu^{\langle H^F_\sigma, a\rangle}) = x\} .$$

From b we know that the arithmetical hierarchy may be constructed, and since F is supposed to be Δ^*-definable, H^F_γ will be.

e If $\gamma = 0$, we always have $1 \in \mathcal{N}_a$ and $|1|_a = 0$.

f If $\gamma = \gamma_0 + 1$ we find the a-notations for γ by

x is an a-notation for $\gamma_0 \Rightarrow 2^x$ is an a-notation for γ .

This exhaust all possibilitites, and we may use theorem 4.4 on definability of inductive definitions. □

Note that we have not assumed or proved that $\langle M_a\rangle_{a\in I}$ is really closed under Harringston's hierarchy. But if $\gamma \in M_a$ and γ has an a-notation e , then $H^F_\gamma \in M_a$, and Harrington's hierarchy up to this point will be Δ^*-definable.

Definition 7.3

a Let $a \in I$. A function f is Δ^*_a over a family $\langle M_b\rangle_{b\in I}$ if f is Δ^* over the family $\langle\langle M_{\langle a,b\rangle}\rangle_{b\in I}, a\rangle$.

b A family $\langle M_b\rangle_{b\in I}$ is strongly impenetrable if for all $a \in I$ and all Δ^*_a-functions f , if f is closed in $\langle M_{\langle a,b\rangle}\rangle_{b\in I}$, then there is a family $\langle N_b\rangle_{b\in I}$, nice relative to a , such that for all b $N_b \subsetneqq M_{\langle a,b\rangle}$, and f is closed in $\langle N_b\rangle_{b\in I}$. □

<u>Remark</u> In most cases, $\langle M_b \rangle_{b\in I}$ will be the spectrum of some theory Θ , and all notions will be relativized to R_Θ . Then $N_b \subsetneqq M_{\langle a,b\rangle}$ actually implies that the ordinals in N_b will be a proper initial segment of the ordinals in $M_{\langle a,b\rangle}$.

In our definition of strongly impenetrability, we claim $\forall b(N_b \subsetneqq M_{\langle a,b\rangle})$. We would have an equivalent definition if we instead claimed $N_0 \subsetneqq N_a$. The proofs we are going to give will work for both possibilities.

To explain the difference between impenetrability and strong impenetrability we notify that: Impenetrability says: It is impossible to run through all of the components of the family. Strong impenetrability says: It is impossible to run through any of the components of the family.

In ω-case, there is only one component, so the two concepts coincide.

<u>Lemma 7.4</u>

Let Θ be a type-k-theory. Let $\langle\langle M_a\rangle_{a\in I}, R_\Theta\rangle$ be the spectrum of Θ . Assume that $\langle M_a\rangle_{a\in I}$ is strongly R_Θ-impenetrable. Let F be Δ^* over $\langle\langle M_a\rangle_{a\in I}, R_\Theta\rangle$. Then, for all $a \in I$, $k+1-sc(F,a) \subseteq M_a$.

<u>Proof</u>

By lemma 7.2 it suffices to prove that when there is an a-notation for σ , then $\sigma \in M_a$. We prove this by induction on σ .

The only new case beyond theorem 4.6 is case <u>vi</u> which introduces recursion in the superjump. So let

$$\sigma = |7^e|_a = \lambda .$$

Assume that the conclusion holds for all ordinals less than λ, and assume that λ is the least ordinal such that

$$\forall b(b \in \mathcal{N} \,\&\, |b| < \lambda \Rightarrow e \in \mathcal{N}_{\langle a,b\rangle} \,\&\, |e|_{\langle a,b\rangle} < \lambda) .$$

Define $g(\alpha) = \mu\beta((\forall b \in \mathcal{N})(|b| < \alpha \Rightarrow |e|_{\langle a,b\rangle} < \beta)$.

By lemma 7.2 and Σ^*-collection, we see that $\langle M_{\langle a,b\rangle}\rangle_{b\in I}$ is closed under g , and that g is Δ_0^* .

Since $\langle M_a\rangle_{a\in I}$ is strongly impenetrable, g will be closed in some $\langle N_b\rangle_{b\in I}$, where $N_0 \subsetneqq M_a$. Let $\lambda' = \mathrm{Sup}\{\mathrm{On} \cap N_0\}$, and let $\lambda'' \in M_a$ be such that $\lambda'' > \lambda'$. Of course, $\lambda \leq \lambda'$ since λ' also has properties <u>a</u> and <u>b</u> in part <u>vi</u> of definition 7.1, and λ was supposed to be the least such ordinal. But then $\lambda < \lambda''$, $\lambda'' \in M_a$ and λ is definable from λ'' and from Harrington's hierarchy. Thus $\lambda \in M_a$ itself. □

<u>Lemma 7.5</u>

Let F be a functional of type $k+2$, let Θ be the Harrington-theory for S^{k+3}, F , and let $\mathrm{Spec}(\Theta) = \langle\langle\langle M_a\rangle_{a\in I}\rangle, R_\Theta\rangle$.

Then R_Θ and $F \cap M(= \bigcup_{a\in I} M_a)$ are Δ^* in each other, and $\langle M_a\rangle_{a\in I}$ is strongly R_Θ-impenetrable.

<u>Proof</u>

By lemmas 7.2 and 7.4 we see that Harrington's hierarchy is Δ^*-definable in $F \cap M$. On the other hand, all information about $F \cap M$ lies in Harrington's hierarchy. Thus F will be Δ^*-definable in this hierarchy, or in R_Θ .

The rest of the theorem will be proved by contradiction. Let f be a counterexample to strong impenetrability. Assume that f is Δ_a^* . Let $\langle N_b\rangle_{b\in I}$ be the least nice closure family for f .

If f is not Δ^* over $\langle N_b\rangle_{b\in I}$ and a, we could use another function f^* defined as

$f^*(x) = \langle f(x),\alpha\rangle$, where α is the least ordinal such that the defining formulas for $f(x)$ are valid inside

$$\langle\langle S_\alpha^{R\,\Theta}(P_{\langle a,b\rangle}\rangle_{b\in I}, a\rangle .$$

(See corollary 5.4 for notation). This new f^* is Δ^*-definable, and Δ^*-definable inside any nice family in which it is closed.

Thus we may assume that f is Δ^* over $\langle N_b\rangle_{b\in I}$ and a.

Then for some b_o, $N_{b_o} = M_{\langle a,b_o\rangle}$. By definition $\langle N_b\rangle_{b\in I}$ will be penetrated by f, so by theorem 6.3 $\langle N_b\rangle_{b\in I}$ is the spectrum of a functional G such that $en(G) \subseteq en(\Theta[a])$. Then G is $\Theta[a]$-computable, by the argument from 3.18. But since G is $\Theta[a]$-computable, there is a $\Theta[a,b_o]$-recursive ordinal greater than all G,b_o-recursive ordinals. Thus $k+1-sc(G,b_o) \in M_{\langle a,b_o\rangle}$. But this is impossible since, by collection $N_{b_o} = \rho''(k+1-sc(G,b_o))$ would be an element of $M_{\langle a,b_o\rangle}$. □

<u>Definition 7.6</u>

Let $\langle M_a\rangle_{a\in I}$ be a nice family relative to R. We say that $\langle M_a\rangle_{a\in I}$ is <u>hyper impenetrable</u> in R if

<u>i</u> $\langle M_a\rangle_{a\in I}$ is strongly R-impenetrable.

<u>ii</u> For all $\Delta^*(R)$-functions f closed in $\langle M_a\rangle_{a\in I}$ there is a strongly R-impenetrable family $\langle N_a\rangle_{a\in I}$ such that

$$\langle N_a\rangle_{a\in I} \subsetneq \langle M_a\rangle_{a\in I}$$

and

$$f \text{ is closed in } \langle N_a\rangle_{a\in I} .$$ □

<u>Remark</u> In the ω-case, hyper-impenetrability is the same as hyper-Mahlo.

<u>Theorem 7.7</u>

Let Θ be a type-k-theory. Then the following two statements are equivalent.

<u>i</u> There is a functional F of type $k+2$ such that Θ is the Harrington-theory of F, S^{k+3}.

<u>ii</u> $\mathrm{Spec}(\Theta)$ is strongly R_Θ-impenetrable, but not R_Θ-hyper-impenetrable.

<u>Remark</u>: Over ω we then obtain. Let α be the ordinal of Θ, then Θ is the Harrington-theory of S^3 and some type-two F if and only if α is R_Θ-Mahlo but not R_Θ-hyper Mahlo.

<u>Proof</u>

<u>i => ii</u> Let F be given as in the assumption, let Θ' be the Harrington-theory of F, S^{k+3}, and let $R_F = R_{\Theta'}$. R_F will then be $\Delta^*(R_\Theta)$. Let $\gamma \in On$ and let $R_F \upharpoonright \gamma = \{\langle a,\beta\rangle \in R_P;\ \beta < \gamma\} = R_F \cap I \times \gamma$.

Let $\mathrm{Spec}(\Theta) = \langle\langle M_a\rangle_{a\in I}, R_\Theta\rangle$.

Recall from section 5 that the hierarchy

$\langle\langle S_\beta^{R_\Theta}(P_a)\rangle_{a\in I}\rangle_{\beta\in On}$ is Δ^*-definable in R_Θ

and that it approximates $\langle M_a\rangle_{a\in I}$. We assume that a $\Delta^*(R_\Theta)$-definition of R_F is given, and, as in the proof of theorem 6.3, we may define a $\Delta^*(R_\Theta)$-function f such that

<u>i</u> f is normal

<u>ii</u> $f(\gamma+1)$ = least β such that the given definition of $R_F \upharpoonright \gamma$ works inside $\langle S_\beta^{R_\Theta}(P_a)\rangle_{a\in I}$.

If f is closed in any strongly impenetrable family $\langle N_a \rangle_{a\in I} \subseteq \langle M_a \rangle_{a\in I}$, we obtain that $\mathrm{Spec}(F, S^{k+3}) \subseteq \langle N_a \rangle_{a\in I}$ componentwise, by lemma 7.4. By assumption then, f cannot be closed in any strongly R_Θ impenetrable proper subfamily of $\langle M_a \rangle_{a\in I}$. But then $\langle M_a \rangle_{a\in I}$ is not R_Θ-hyper impenetrable.

That $\mathrm{Spec}(\Theta)$ is strongly R_Θ-impenetrable follows from theorem 3.17 and lemma 7.5.

<u>ii => i</u> Let f be a counterexample to hyper-impenetrability. We may assume that f is only defined for ordinal numbers. For each γ, define $R_\gamma = \{\langle a,b,\gamma \rangle$ a and b are Θ-computations and $a \leq_\Theta b$ and $|b|_\Theta < \gamma\}$. Let $R = \bigcup_{\gamma\in On} R_\gamma$. As in the proof of theorem 6.3 R and R_Θ are Δ^* in each other. Since f is closed in no R_Θ-strongly impenetrable proper subfamily of $\langle M_a \rangle_{a\in I}$, this family will be the least family strongly impenetrable in R.

From the proof of theorem 6.1, we recall the following definition of F:

$$F(A) = \begin{cases} 1 & \text{if } A \text{ is a code for an element in } R \\ 0 & \text{otherwise} \end{cases}$$

As in that proof, R and F will be Δ^* in each other over any nice family. Thus $\langle M_a \rangle_{a\in I}$ is the least strongly impenetrable family relative to F.

By lemma 7.4 we then obtain that $\mathrm{Spec}(F, S^{k+3}) = \mathrm{Spec}(\Theta)$, and R_F and R_Θ will be Δ^* in each other.

This ends the proof of the theorem.

$\square$

APPENDIX

In this Appendix we have collected detailed proofs for a number of technical results from sections 2 and 3 .

A.1 PROOF OF LEMMA 2.2

We claim that A is a code if and only if

i A consists of pairs

ii A is well-founded

$$(\forall a)((\forall i)\langle (a)_{i+1},(a)_i\rangle \in A \Rightarrow \exists j\ \forall i \geq j\ \langle (a)_i,(a)_{i+1}\rangle \in A)$$

iii $\langle 0,a\rangle$ will always be a minimal element

$$(\forall a\ \forall b)(\langle b,\langle 0,a\rangle\rangle \in A \Rightarrow b = \langle 0,a\rangle)$$

iv Extensionality except for urelements

$$\begin{aligned}(\forall a,b \in \operatorname{dom} A)[\forall c\ (\langle c,a\rangle \in A \ \&\ \langle a,c\rangle \notin A \\ &\Leftrightarrow \langle c,b\rangle \in A \ \&\ \langle b,c\rangle \notin A) \\ &\Leftrightarrow ((\langle a,b\rangle \in A \ \&\ \langle b,a\rangle \in A) \\ &\quad \text{or both } a \text{ and } b \text{ are on the} \\ &\quad \text{form } \langle 0,d\rangle \text{ and } \langle 0,e\rangle)] .\end{aligned}$$

v Maximal elements exist and are equivalent.

$$(\exists a)(\forall b)(\langle a,b\rangle \in A \Rightarrow \langle b,a\rangle \in A)$$

& All maximal elements (like a above) are equivalent.

vi $(\forall\langle a,b\rangle \in A)(\langle a,a\rangle \in A \ \&\ \langle b,b\rangle \in A)$.

Obviously all codes satisfy i-vi. By induction on the height of the well-founded relation A , we prove that any A satisfying i-iv is a code.

Say $a \simeq b \Leftrightarrow \langle a,b\rangle \in A \ \&\ \langle b,a\rangle \in A$.

Let A_α be the elements of rank α in A . To each $a \in \operatorname{dom} A$, we define a set x_a , and prove by induction on rank a , that

* $\quad x_a = x_b \Leftrightarrow a \simeq b$.

$\underline{\alpha = 0}$ If $a = \langle 0,b\rangle$, then let $x_a = b$.

If $a \neq \langle 0,b\rangle$ for all b , let $x_a = \emptyset$.

In the first case, we obviously by <u>iii</u> and <u>vi</u> have

$x_a = x_b \Leftrightarrow a \simeq b$.

In the second case, we use extensionality.

$\underline{\alpha > 0}$ Assume * holds for rank $c < \alpha$, and let a be of rank α.

Let $x_a = \{x_b;\ \langle b,a\rangle \in A \ \&\ \langle a,b\rangle \notin A\}$.

$$x_a = x_b \Leftrightarrow (\forall y)(y \in x_a \Leftrightarrow y \in x_b) \Leftrightarrow (\forall c)(x_c \in x_a \Leftrightarrow x_c \in x_b)$$

$$\Leftrightarrow \forall c(\langle c,a\rangle \in A \ \&\ \langle a,c\rangle \notin A \Leftrightarrow \langle c,b\rangle \in A \ \&\ \langle b,c\rangle \notin A)$$

$$\Leftrightarrow a \simeq c \ .$$

The last equivalence uses iv.

Let a be maximal. Then A is a code for x_a .

A.2 PROOF OF LEMMA 2.3

<u>a</u> Let
$$A_1 = \{\langle a,b\rangle;\ (\exists a_1,b_1)(a = \langle 0,a_1\rangle \ \&\ b = \langle 1,b_1\rangle \ \&\ \langle a,b_1\rangle \in A)$$
$$\vee\ (\exists a_1,b_1)(a = \langle 1,a_1\rangle \ \&\ b = \langle 1,b_1\rangle \ \&\ \langle a,b\rangle \in A$$
$$\&\ \forall c(a_1 \neq \langle 0,c\rangle \ \&\ b_1 \neq \langle 0,c\rangle))$$
$$\vee\ (\exists a_1,b_1)(a = \langle 0,a_1\rangle \ \&\ b = \langle 0,b_1\rangle \ \&\ \langle a,b\rangle \in A)\} \ .$$

What we have done here is to create a new code for x .
If a set $y \in TC(\{x\})$, not being an urelement, is coded by a in A , we let it be coded by $\langle 1,a\rangle$ in A_1 . When we need a new urelement as a code, we now have a large supply to take it from.

Identifying a function by it's graph, we see

$$f = \{\{\{a\} , \{a,y\}\} ; f(a) = y\}$$

Obviously $f \notin TC(\{x\})$, but some of the sets $\{\{a\} , \{a,y\}\}$, $\{a\}$ or $\{a,y\}$ may be elements of $TC(\{x\})$.

The actual a's can, however, be picked out by a first order formula. To give an example we will do it for $\{a,y\}$.

$\{a,y\} \in TC(\{x\}) \Leftrightarrow$ it has a code $b \in \operatorname{dom} A_1$.

Recall that a codes y . Thus, if such b exists, we must have

$$\langle\langle 0,a\rangle,b\rangle \in A_1 \qquad (a \in \{a,y\})$$
$$\langle b,\langle 0,a\rangle\rangle \notin A_1$$

$\langle a,b\rangle \in A_1$ and $\langle b,a\rangle \notin A_1$ $\quad (y \in \{a,y\})$.
For all c , if $\langle c,b\rangle \in A_1$ and $\langle b,c\rangle \notin A_1$, then $c = a$ or $c = \langle 0,a\rangle$ (Only a and y are elements of $\{a,y\}$) .
These properties of b characterizes the codes for $\{a,y\}$.
Now, let $\langle 5,0\rangle$ code f .

$\langle 4,a\rangle$	codes	$\{\{a\} , \{a,y\}\}$	if the set has not a code in A_1
$\langle 3,\langle a,1\rangle\rangle$	codes	$\{a\}$	– " –
$\langle 3,\langle a,2\rangle\rangle$	codes	$\{a,y\}$	– " –

The definition of a code for f is now simple. However, the formula is long and of no real interest.

<u>b</u> Let A_1 and A_2 be codes for α_1 and α_2 .
Let $B_1 = \{\langle\langle 1,a\rangle , \langle 1,b\rangle\rangle ; \langle a,b\rangle \in A_1\}$

$B_2 = \{\langle\langle 2,a\rangle , \langle 2,b\rangle\rangle ; \langle a,b\rangle \in A_2\}$

Let $\langle a,b\rangle \in C \Leftrightarrow \langle a,b\rangle \in B_1 \vee \langle a,b\rangle \in B_2$
$\vee\ a \in \operatorname{dom} B_1 \ \&\ b \in \operatorname{dom} B_2$.

C will be a code for $\alpha_1 + \alpha_2$. □

A.3 PROOF OF LEMMA 2.5 a

The proof is by induction on the structure of φ

i $\quad \varphi(x,y) = x \in y$

Let

$$f_\varphi(x,y) = \cup\ ((y \setminus (y \setminus \{x\})) \times \{y\}) \ .$$

$\cup\ ((y \setminus (y\setminus\{x\})) \times \{y\}) = \emptyset$

$\iff y \setminus (y\setminus\{x\}) \times \{y\} = \{\emptyset\} \vee (y \setminus (y\setminus\{x\})) \times \{y\} = \emptyset$

$\iff y \setminus (y\setminus\{x\}) = \emptyset$, since $\emptyset$ is not a pair

$\iff x \notin y$.

Now, let $x \in y$. Then $y \setminus (y\setminus\{x\}) = \{x\}$

$\{x\} \times \{y\} = \{\langle x,y\rangle\}$, and $\cup \{\langle x,y\rangle\} = \langle x,y\rangle$.

ii To avoid confusion with set pairing,, we here let $\prec , \succ$ denote the pairing on I .

$\varphi(x,y,z) = \prec x,y\succ = z$.

Let $f_\varphi = \langle x,y,z\rangle \setminus \cup\ (\{\langle x,y,z\rangle\} \setminus \{\langle x,y, \prec x,y\succ \rangle\})$.

Then

$\prec x,y\succ = z \Rightarrow \{\langle x,y,z\rangle\} = \{\langle x,y,\prec x,y\succ\rangle$

and $f_\varphi(x,y,z) = \langle x,y,z\rangle$

$\prec x,y\succ \neq z \Rightarrow \{\langle x,y,z\rangle\} \cap \{\langle x,y,\prec x,y\succ\rangle\} = \emptyset$

and $f_\varphi(x,y,z) = \emptyset$.

By the same method we verify the lemma for

$$\varphi(x,i,z) \iff (x)_i = z \ , \ \varphi(a,x) \iff c_i(a) = x$$

and $\varphi(x,y,z) \iff ev^{(i)}(x,y) = z$.

Note that set pairing is rudimentary.

<u>iii</u> $\varphi(\vec{x}) = \neg\psi(\vec{x})$

Let $f_\varphi(\vec{x}) = \langle\vec{x}\rangle \setminus f_\psi(\vec{x})$

<u>iv</u> $\varphi(\vec{x}) = \psi_1(\vec{x}) \vee \psi_2(\vec{x})$

$f_\varphi(\vec{x}) = f_{\psi_1}(\vec{x}) \cup f_{\psi_2}(\vec{x}) = \cup\{f_{\psi_1}(\vec{x}), f_{\psi_2}(\vec{x})\}$

The verifications are trivial in cases <u>iii</u> and <u>iv</u> .

<u>v</u> $\varphi(y,x_2,\ldots,x_n) = (\exists x \in y)\,\psi\,(x,x_2,\ldots,x_n)$.

Let h be the rudimentary function defined by

$$h(\{y\} \times \{x\}) = \begin{cases} \{\langle y,x_2,\ldots,x_n\rangle\}\ , & \text{when } x = \langle x_1,\ldots,x_n\rangle \text{ for some } x_1\ , \\ \emptyset\ , & \text{otherwise.} \end{cases}$$

Let $f_\varphi(y,x_2,\ldots,x_n) = \bigcup_{x_1\in y} h(\{y\}\times\{f_\psi(x_1,\ldots,x_n)\})$.

Then

$$\exists x_1 \in y\psi(x_1,\ldots,x_n) \Rightarrow (\exists x_1 \in y)(f_\psi(x_1,\ldots,x_n) = \langle x_1,\ldots,x_n\rangle)$$

$$\Rightarrow (\exists x_1 \in y)\, h\,(\{y\}\times\{f_\psi(x_1,\ldots,x_n)\} = \{\langle y,x_2,\ldots,x_n\rangle\}\ .$$

Since $h(\{y\}\times\{f_\psi(x_1,\ldots,x_n)\})$ is either $\langle y,x_2,\ldots,x_n\rangle$ or $\emptyset$, we have

$$f_\varphi(y,x_2,\ldots,x_n) = \langle y,x_2,\ldots,x_n\rangle\ .$$

If $f_\psi(x_1,\ldots,x_n) = \emptyset$ for all $x_1 \in y$, we see that

$$f_\varphi(y,x_2,\ldots,x_n) = \emptyset\ .$$

<u>vi</u> $\varphi(x) = x_i \in F$

$$f_\varphi = \cup\,(\{\langle x_1,\ldots x_{i-1}\rangle\} \times F \cap \{x_i\} \times \{\langle x_{i+1},\ldots,x_n\rangle\}\ .$$

This completes the proof.

A.4 PROOF OF LEMMA 2.7

"To be a sequence of codes" is first order definable. Also condition iii in definition 2.6 is 1. order. So, let $A_1, \ldots, A_n$ be a sequence of codes satisfying iii. We claim that $A_1, \ldots, A_n$ satisfies i and ii in definition 2.6 if and only if

$$\begin{aligned}
&(\forall c_i, c_j)(c_i \in \operatorname{dom} A_i \ \&\ c_j \in \operatorname{dom} A_j \\
&\qquad \&\ \forall d(\langle d, c_i\rangle \in A_i \ \&\ \langle c_i, d\rangle \notin A_i \\
&\qquad\qquad \Longleftrightarrow \langle d, c_j\rangle \in A_j \ \&\ \langle c_j, d\rangle \notin A_j) \\
&\qquad \Rightarrow c_i \text{ and } c_j \text{ are both codes for urelements.} \\
&\qquad\quad \text{or } \langle c_i, c_j\rangle \in A_i \cap A_j \ \&\ \langle c_j, c_i\rangle \in A_i \cap A_j)) .
\end{aligned}$$

Obviously any compatible sequence will satisfy this, and compatibility now follows by a simple induction.

A.5 PROOF OF LEMMA 2.9

We solve the problem pointed out in the remark in this way:

We may effectively enumerate all rudimentary functions $\{f_n\}_{n\in\omega}$ such that when f_i is a subfunction of f_j, then $i < j$.

Then, at each level m in the proof,

i Assume that $\{f_i(x_1, \ldots, x_n)\}_{i<m}$ are (dummy) arguments in f_m.

ii For all new codes, $\langle k, a\rangle$, to be constructed, choose $k \in \{m\} \times \omega$.

We obtain by this a stronger lemma, viz.

$$f_i^*(A_1, \ldots, A_n) \text{ is compatible to } f_j^*(A_1, \ldots, A_n)$$

for all i, j.

We proceed to the details of the proof.

Note that the function $h(A) = \operatorname{dom} A$ is a rudimentary function.

We say that a is a main member in A_j if a codes x_j.

<u>i</u> $\quad f(x_1,\ldots,x_n) = x_i \qquad f^* = f$

<u>ii</u> $\quad f(x_1,\ldots,x_n) = x_i \setminus x_j$

Let

$$B_k = \{\langle a,b\rangle \in A_k;\ \langle b,a\rangle \notin A_k\}$$

B_k is the 'strict' code for x_k and is first order definable over $\operatorname{dom} A_k$.

Let

$$g_1(A_k) = \{(a)_o;\ a \in B_k\} = \bigcup_{a\in B_k} \{(a)_o\}$$

$g_1(A_k)$ consists of all non-main members of A_k. Let

$$g_2(A_k) = \operatorname{dom} A_k \setminus g_1(A_k)$$

$g_2(A_k)$ consists of all main members of A_k. Let

$$g_3(A_k) = \{(a)_o;\ a \in B_k \ \&\ (a)_1 \in g_2(A_k)\}$$

$g_3(A_k)$ consists of all codes for the elements of x_k.

Let

$$f_1(A_1,\ldots,A_n) = g_3(A_i) \setminus g_3(A_j)$$

i.e. the codes for the elements in $x_i \setminus x_j$. Here we use that A_i and A_j are compatible.

Let f' be the function from lemma 2.8. Let

$$f_2(A_1,\ldots,A_k) = \bigcup_{\substack{a\in f_1(A,\ldots,A_n)\\ k\le n}} f'(A_k,a).$$

f_2 gives us the original code restricted to $TC(x_i \setminus x_j)$.

Now we must split our construction into two cases:

1. $x_i \setminus x_j \in TC(\{x_k\})$, for some k

2. $x_i \setminus x_j \notin TC(\{x_k\})$, for any k .

We prove that we can decide effectively between these cases. We have

a codes $x_i \setminus x_j$ in A_k if and only if

$$(\forall b \in I)(\langle b,a\rangle \in B_k \Rightarrow (b \in \text{dom } f_2(A_1,\dots,A_n)$$
$$\& (\forall c \in I)(\langle b,c\rangle \in f_2(A_1,\dots,A_n) \Rightarrow \langle c,b\rangle \in f_2(A_1,\dots,A_n))))$$

Now, let

$$f^*(A_1,\dots,A_n) = \{\langle a,b\rangle \in f_2(A_1,\dots,A_n)$$
$$\vee (b \text{ codes } x_i \setminus x_j \text{ in some } A_k$$
$$\& \langle a,b\rangle \in A_k)$$
$$\vee ((\forall c)(c \text{ does not code } x_i \setminus x_j \text{ in any } A_k)$$
$$\& b = \langle m,0\rangle \& (a = \langle m,0\rangle \vee a \text{ codes an}$$
$$\text{element of } x_i \setminus x_j)))\} .$$

Here m is the least number m such that for no a, k, $\langle m,a\rangle \in \text{dom } A_k$.

It should easily be seen that this definition works.

<u>iii</u> $f(x_1,\dots,x_n) = \{x_i,x_j\}$.

a is a code for $\{x_i,x_j\}$ in A_k

$$\Leftrightarrow (\forall b)(((\langle b,a\rangle \in A_k \Leftrightarrow \langle a,b\rangle \in A_k)$$
$$\vee b \in g_2(A_i) \vee b \in g_2(A_j))$$
$$\& (\exists b_1,b_2)(b_1 \in g_2(A_i) \& b_2 \in g_2(A_j) \& \langle a,b_1\rangle \notin A_k$$
$$\& \langle a,b_2\rangle \notin A_k) .$$

Now let

$$f^*(A_1,\ldots,A_n) =$$

$$A_i \cup A_j \cup \{\langle a,b\rangle ;\ ((a \in g_2(A_i) \vee a \in g_2(A_j))\ \&\ b \text{ is a code for } \{x_i,x_j\} \text{ in some } A_k)$$
$$\vee\ (a \text{ and } b \text{ are both codes for } \{x_i,x_j\} \text{ in some } A_k)$$
$$\vee\ (\forall c)(c \text{ is not a code for } \{x_i,x_j\} \text{ in any } A_k\ \&\ b = \langle m,0\rangle$$
$$\&\ (a= \langle m,0\rangle \vee a \in g_2(A_i) \vee a \in g_2(A_j)))\}.$$

Also here m denotes the least number m such that $\langle m,a\rangle \notin \operatorname{dom} A_k$ for any $a \in I$, $k \leq n$.

<u>iv</u> $f(x_1,\ldots,x_n) = \bigcup_{y\in x_1} h(y,x_2,\ldots,x_n)$

Let f' be the function from lemma 2.8 and let

$$f_4(A_1,\ldots,A_n) = \bigcup_{a\in g_3(A_1)} h^*(f'(A_1,a),\ A_2,\ldots,A_n))$$

As in the previous cases, we may effectively split between the cases: Is $f(x_1,\ldots,x_n)$ coded in some A_k or not? Then we may give a proper definition of f^* .

<u>v</u> $f(x_1,\ldots,x_n) = h(g_1(x_1,\ldots,x_n),\ldots,g_m(x_1,\ldots,x_n))$

$f^*(A_1,\ldots,A_n) = h^*(g_1^*(A_1,\ldots,A_m),\ldots,g_m^*(A_1,\ldots,A_m))$.

<u>vi</u> The code of an element a is $\langle 0,a\rangle$.

The code of an element $\langle a,b\rangle$, will then be $\langle 0,\langle a,b\rangle\rangle$.

If $f(a,i) = (a)_i$, then

$f^*(\langle 0,a\rangle,\ \langle 0,i\rangle) = \langle 0,(a)_i\rangle$

Analogous for c_i and $ev^{(i)}$.

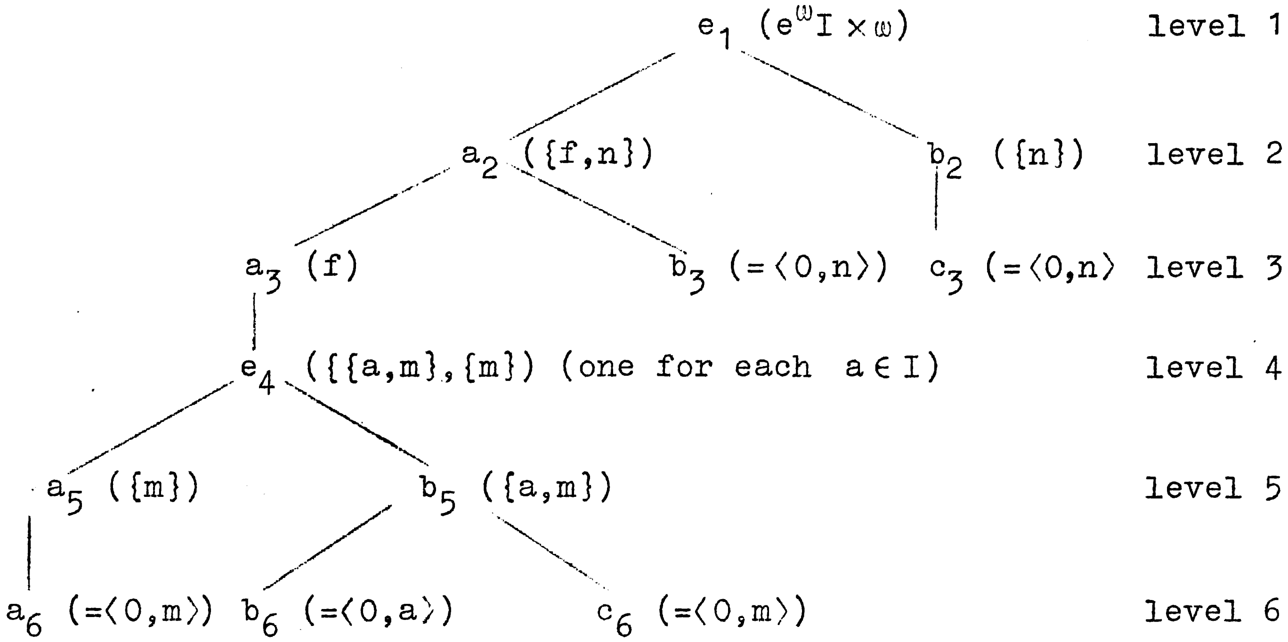
<u>vii</u> $f(x_1,\ldots,x_n) = x_i \cap F$.

We use the following pairing of sets

$$F = \{\{\{f,n\},\ \{n\}\};\ F(f) = n\}$$

$$f = \{\{\{a,m\},\ \{m\}\};\ f(a) = m\}$$

The code structure for the elements of F is as follows, parentheses indicating what is coded:



Obviously, if $e \in dom(A_i)$, we may check by a first order formula if e has a decoding like the one above. Also the levels and subcode relation, are first order definable.

Now we are going to pick out those e_1 coding elements of F. If e_1 is given as in the diagram, the function

$$f_{e_1} = \{\langle (b_6)_1, (a_6)_1 \rangle ; a_6 \text{ and } b_6 \text{ are subcodes of the same } e_4 \text{ which again is a subcode of } e_1\}$$

is first order definable, uniformly in e_1

e_1 codes an element of F if $F(f_{e_1}) = (c_3)_1$. This too is first order in F.

Now we must check if $F \cap x_i$ is coded in some A_k, but this is done exactly as in case ii.

Here F was of type $k+2$. Note that functionals of type $\geq k+2$ have too large cardinality to have a code.

This concludes the proof.

A.6 PROOF OF LEMMA 2.10

We prove the lemma by induction on the structure of Ψ. Let S be the universe of compatible sequences.

i $\quad \Psi = x_i \in x_j$

$$\vec{A} \in A_\Psi \iff (\exists a \in g_3(A_j))(A_i = f'(a,A_j)) \;\&\; \vec{A} \in S$$

where g_3 and f' are as in the proof of lemma 2.9.

ii $\quad \Psi = \Phi_1 \vee \Phi_2$. $A_\Psi = A_{\Psi_1} \cup A_{\Phi_2}$

iii $\quad \Psi = \neg\Phi$. $A_\Psi = S \setminus A_\Phi$

iv $\quad \Psi = \exists x \in y \Phi$

$$(B_1,\vec{A}) \in A_\Psi \iff (\exists a \in g_3(B_1))((f'(a,B_1), A_1,\ldots,A_n) \in A_\Phi)$$

where $\langle B_1, \vec{A} \rangle \in S$.

v $\quad \Psi = x \in F$. See last part of proof of lemma 2.9.

$\square$

A.7 PROOF OF LEMMA 3.6

We will prove that $x \in M_a$ if and only if x has a code in M_a .

i $\quad$ Let $A \in M_a$ be a code for x . Since $A \in M_a$, there must be a code for A in $\mathcal{Y}_a$. But then A itself will be in $\mathcal{Y}_a$, and $x \in M_a$.

ii $\quad$ Let $x \in M_a$. Then x has a code A in $\mathcal{Y}_a$, It is not hard to define a code for A in $\mathcal{Y}_a$, and thus $A \in M_a$.

$\square$

A.8 PROOF OF LEMMA 3.7 ii

Let $A_1, \ldots, A_n$ be a sequence of codes in M_a , Without loss of generality we may assume that $n = 2$, that each A_i is compatible with itself and that

$$\operatorname{dom} A_1 \cap \operatorname{dom} A_2 \subseteq \{0\} \times I \text{ .}$$

By induction on the rank of the codes, we are going to identify codes coding the same set. Formally, we use the fix-point theorem:

$$
G(g,\langle a,b\rangle) = \left\{ \begin{array}{lll}
 & 1 & \text{if } \ a \in \mathrm{dom}(A_1 \cup A_2) \,\&\, b \in \mathrm{dom}(A_1 \cup A_2) \ \& \\
(1) & & (\langle a,b\rangle \in A_1 \vee \langle a,b\rangle \in A_2 \\
(2) & & \vee\ (\exists i\ \exists c)(\langle c,b\rangle \in A_i \,\&\, \langle b,c\rangle \notin A_i \\
 & & \qquad\qquad\qquad\qquad \& \ g(\langle a,c\rangle) = g(\langle c,a\rangle) = 1) \\
(3) & & \vee\ \ a \text{ and } b \text{ does not code urelements} \\
 & & \quad \& \ \ a \in \mathrm{dom}\, A_{i_1} \,\&\, b \in \mathrm{dom}\, A_{i_2} \\
 & & \quad \& \ \ (\forall c_1\ \exists c_2)(\langle c_1,a\rangle \in A_{i_1} \,\&\, \langle a,c_1\rangle \notin A_{i_2} \\
 & & \qquad\qquad \Rightarrow \langle c_2,b\rangle \in A_{i_2} \,\&\, \langle b,c_2\rangle \notin A_{i_2} \\
 & & \qquad\qquad \& \ g(\langle c_1,c_2\rangle) = g(\langle c_2,c_1\rangle) = 1 \\
 & & \quad \& \ \ (\forall c_2\ \exists c_1)(\langle C_2,b\rangle \in A_{i_2} \,\&\, \langle b,c_2\rangle \notin A_{i_1} \\
 & & \qquad\qquad \Rightarrow \langle c_1,a\rangle \in A_{i_1} \,\&\, \langle a,c_2\rangle \notin A_{i_1} \\
 & & \qquad\qquad \& \ g(\langle c_1,c_2\rangle) = g(\langle c_2,c_1\rangle) = 1 \\
 & 0 & \text{otherwise, where all occurences of} \\
 & & \neg(g(\langle d,e\rangle) = 1) \ \text{ in the negated formula is} \\
 & & \text{replaced by } \ g(\langle d,e\rangle) = 0\ .
\end{array}\right.
$$

The numbers in parentheses indicates the numbers of the disjuncts.

Let g be a fix-point for G. We prove that g is the characteristic function for a set A having the following property:

$\langle a,b\rangle \in A$ if and only if a codes a set x_a in A_1 or in A_2, b codes a set x_b in A_1 or in A_2 and $x_a \in x_b$ or $x_a = x_b$.

By the first line in the definition of G, we see that if $g(\langle a,b\rangle) = 1$, then x_a and x_b exist.

We prove the claim by induction of the following ordering

$(a,b) \leq (c,d)$ if $\max \mathrm{rank}(x_a,x_b) \leq \max \mathrm{rank}(x_c,x_d)$
If $\max \mathrm{rank}(x_a,x_b) = \max \mathrm{rank}(x_c,x_d)$, then
$(a,b) \leq (c,d)$ if $\min \mathrm{rank}(x_a,x_b) \leq \min \mathrm{rank}(x_c,x_d)$.

Max rank$(x_a, x_b) = 0$. There are several possibilities:

<u>i</u> x_a and x_b are urelements and $x_a = x_b$.
Then disjunct 1 holds.

<u>ii</u> $x_a = \emptyset$ and x_b is an urelement, or they are different urelements. Then the last disjunct is obviously false, the second says that b is not minimal in A_i and the first is again obviously wrong.

<u>iii</u> $x_a = x_b = \emptyset$. The premisses in the two implications in the third disjunct will be false for all C_1, C_2 , and thus this disjunct has to be correct.

Next let max(rank x_a, rank x_b) > 0 , and assume the IH . Again we have three cases.

<u>i</u> $x_a = x_b$. Then the third disjunct says:
"For all elements in x_a we may find it in x_b and vice versa" by the IH.
This will be correct; and $g(\langle a,b\rangle) = 1$.

<u>ii</u> $x_a \in x_b$. By IH the second disjunct says that
"There is an element in x_b that equals x_a".
This will then be correct, and $g(\langle a,b\rangle) = 1$.

<u>iii</u> If $x_a \neq x_b$, we may find a counterexample to disjunct 3 , and the second disjunct will never be fulfilled. The first will obviously be false, and then $g(\langle a,b\rangle) = 0$.

From A, A_1 and A_2 it is now trivial to define compatible codes for x_1 and x_2 .

□

A.9 PROOF OF LEMMA 3.13

Let φ be Δ_0 in R and assume that $M_a \models \forall x \, \exists y \, \varphi(x,y)$. As in the proof of lemma 3.12, we may prove that the set

$$C_0 = \{\langle e,e'\rangle \; ; \; e \text{ and } e' \text{ are a-indices for codes } C_e \text{ and } C_{e'} \text{ such that } \varphi(\rho(C_e,C_{e'}))\}$$

is semicomputable, uniformly in a.

There is a function g, computable in $\circledast[a]$, such that whenever e is an a-index for a code, then

$$\langle e,g(e)\rangle \in C_0 \, .$$

Let $f(o) = e$ for some a-index e for a code, and let

$f(n+1) = g(f(n))$. Then f is $\circledast[a]$-computable, and $\langle f(n)\rangle_{n\in\omega}$ is a sequence for a-indices for codes for sets $\langle x_i\rangle_{i\in\omega}$ such that $\forall i \, \varphi(x_i,x_{i+1})$.

Again we may assume that the codes are compatible, and that we may effectively find a "new" urelement outside the domains of the given codes, that we may use as a code for $\langle x_i\rangle_{i\in\omega}$. The definition of a code for $\langle x_i\rangle_{i\in\omega}$ is now fairly straight forward and is omitted here.

$\square$

A.10 PROOF OF LEMMA 3.14

Let $A \in M_a$ be a code for a clean set x, and let $B \in M_a$ be a code for an element $y \in x$.

Using the fix-point theorem, we may find the part of A coding the same set as B does. Thus the immediate subcodes of A are themselves $\circledast[a]$-computable.

The set

$$C = \{\langle b,e\rangle \; ; \; b \in g_3(A) \text{ and } e \text{ is an a-index for the subcode of } A \text{ with main member } b\}$$

will be $\Theta[a]$-semi-computable. Let f_o be a $\Theta[a]$-computable function such that

$$(\forall b \in g_3(A))(\langle b, f_o(b)\rangle \in C)$$

and let

$$f(b) = \mu n(\exists c \in g_3(A))(\langle b,c\rangle \in A \,\&\, f_o(c) = n) \ .$$

Then f is computable, defined on $g_3(A)$, and if b and c codes the same set, then $f(b) = f(c)$ will be an index for the code for that set. Now it is a simple task to define a code for an enumeration of x. $\square$

A.11 PROOF OF LEMMA 3.16

a We prove the lemma by induction on the complexity of Δ_o-formulas.

i $Q = M \setminus Q_1$ and Q_1 is Δ^*, then Q is Δ^* by definition.

ii Assume $Q = Q_1 \cup Q_2$, and let $\varphi_1, \varphi_2, \psi_1$ and ψ_2 be Δ_o-formulas such that

$$\vec{x} \in Q_i \iff \exists y \in M\ \varphi_i(\vec{x},y) \iff \exists y \in M_{\vec{x}}\ \varphi_i(\vec{x},y)$$
$$\Rightarrow \forall y \in M_{\vec{x}}\ \psi_i(\vec{x},y) \Rightarrow \forall y \in M\ \psi_i(\vec{x},y) \ ,$$

where $M = \bigcup_{a\in I} M_a$.

The Σ-form for Q will be

$$\vec{x} \in Q \iff \exists y(\varphi_1(\vec{x},y) \vee \varphi_2(\vec{x},y))$$
$$\iff (\exists y \in M_{\vec{x}})(\varphi_1(\vec{x},y) \vee \varphi_2(\vec{x},y))$$

The Π-form for Q will be

$$\vec{x} \in Q \iff \forall y(\psi_1(\vec{x},y)) \vee \forall y(\psi_2(\vec{x},y))$$

$$\iff \forall y(\psi_1(\vec{x},(y)_o) \vee \psi_2(\vec{x},(y)_1))$$

$$\vec{x} \in Q \iff (\forall y \in M_{\vec{x}})(\psi_1(\vec{x},y)) \vee (\forall y \in M_{\vec{x}})(\psi_2(\vec{x},y))$$

$$\iff (\forall y \in M_{\vec{x}})(\psi_1(\vec{x},(y)_o) \vee \psi_2(\vec{x},(y)_1)$$

<u>iii</u> $\langle x,\vec{x}\rangle \in Q \iff (\exists y \in x)(\langle y,\vec{x}\rangle \in Q_1)$

Let φ_1 and ψ_1 be as above. Then

$$\forall y \in x \; \exists z \in M_{y,\vec{x}} \; (\subseteq M_{x,y,\vec{x}})(\varphi(y,\vec{x},z) \vee \neg\psi(y,\vec{x},z)) .$$

By Σ^*-collection there is a set $u \in M_{x,\vec{x}}$ such that

(*) $(\forall y \in x)(\exists z \in u)(\varphi(u,\vec{x},z) \vee \neg\psi(y,\vec{x},z))$.

We claim that the following formulas define Q and has the required absoluteness properties:

(I) $(\exists u \in M_{x,\vec{x}})(\forall y \in x \; \exists z \in u \; (\varphi(y,\vec{x},z) \vee \neg\psi(y,\vec{x},z))$
$\& \; \exists y \in x \; \exists z \in u \; \varphi(y,\vec{x},z))$

(II) $(\forall u \in M_{x,\vec{x}})(\forall y \in x \; \exists z \in u \; (\varphi(y,\vec{x},z) \vee \neg\psi(y,\vec{x},z))$
$\Rightarrow \exists y \in x \; \exists z \in u \; \varphi(y,\vec{x},z))$.

To prove the claim there are four implications to establish:

1. Assume $(\exists y \in x)(\langle y,\vec{x}\rangle \in Q_1)$

1.1 Validity of (I): We use the u from (*), and pick one y in x satisfying $(\langle y,\vec{x}\rangle \in Q_1)$. Since we cannot have both $\exists z \; \varphi(y,\vec{x},z)$ and $\exists z \; \neg\psi(y,\vec{x},z)$, the actual z from (*) must satisfy $\varphi(y,\vec{x},z)$.

1.2 Validity of (II): Let $u \in M$ be arbitrary satisfying

$$\forall y \in x \, \exists z \in u \, \varphi(y,\vec{x},z) \vee \neg \psi(y,\vec{x},z)) .$$

Let y be as above. By the same argument, the actual z must satisfy $\varphi(y,\vec{x},z)$.

2.1 Now, let $u \in M$ be an arbitrary set satisfying (I) (i.e. we do not claim that $u \in M_{x,\vec{x}}$ but that u satisfies the rest of (I)). Then there is a $y \in x$ and a $z \in u$ such that $\varphi(y,\vec{x},z)$. But then $\exists z \, \varphi(y,\vec{x},z)$ holds for some $y \in x$, and $\langle x,\vec{x}\rangle \in Q$.

2.2 At last, assume that for all $u \in M_{x,\vec{x}}$, (II) holds. Let then u come from (*). Then the premise in (II) is satisfied and we may proceed as in 2.1.

From the proof we see that the Δ_0 formula φ we constructed has the properties:

$$\exists u \, \varphi \Rightarrow \vec{x} \in Q \qquad \vec{x} \in Q \Rightarrow \exists u \in M_{x,\vec{x}} \, \varphi$$

$$\forall u \in M_{x,\vec{x}} \, \varphi \Rightarrow \vec{x} \in Q \qquad \vec{x} \in Q \Rightarrow \forall u \, \varphi$$

The absoluteness properties then follow trivially.

.b Let φ and ψ be Δ_0-formulas such that

$$\vec{x} \in P \Leftrightarrow \exists y \in M_{\vec{x}} \, \varphi(y,\vec{x}) \Leftrightarrow \forall y \in M_{\vec{x}} \, \psi(y,\vec{x})$$

$$\Leftrightarrow \exists y \, \varphi(y,\vec{x}) \Leftrightarrow \forall y \, \psi(y,\vec{x})$$

Let $\vec{x} \in TC(M_b)$. Then for some $u \in M_b$, $\vec{x} = x_1,\dots,x_n$ are all elements of $TC(u) \in M_b$. Assume that u is transitive. We know

$$\forall \vec{x} \in u^n \, \exists y \in M_{\vec{x}} \, (\varphi(y,\vec{x}) \vee \neg \psi(y,\vec{x})) .$$

Again we find a $v \in M_b$ such that

$$\forall \vec{x} \in u^n \; \exists y \in v \; (\varphi(y,\vec{x}) \vee \neg \psi(y,\vec{x})) \; .$$

Thus, when $\vec{x} \in TC(M_b)$ we have

$$\vec{x} \in P \iff \exists y \in TC(M_b)\, \varphi(y,\vec{x}) \iff \forall y \in TC(M_b)\, \psi(y,\vec{x})$$

Our definition will then be

$$\vec{x} \in P \iff \exists v \; \exists y \in v \; \varphi(y,\vec{x})$$

$$\iff \forall v \; \forall y \in v \; \psi(y,\vec{x})$$

over M_b .

$\square$

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